

GENERATING FUNCTIONS FOR HECKE OPERATORS

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ABSTRACT. Fix a prime N , and consider the action of the Hecke operator T_N on the space $\mathcal{M}_\kappa(SL(2, \mathbf{Z}))$ of modular forms of full level and varying weight κ . The coefficients of the matrix of T_N with respect to the basis $\{E_4^i E_6^j \mid 4i + 6j = \kappa\}$ for $\mathcal{M}_\kappa(SL(2, \mathbf{Z}))$ can be combined for varying κ into a generating function F_N . We show that this generating function is a rational function for all N , and present a systematic method for computing F_N . We carry out the computations for $N = 2, 3, 5$, and indicate and discuss generalizations to other spaces of modular forms.

1. INTRODUCTION

In this article, we introduce a generating function for the action of a Hecke operator on the spaces of elliptic modular forms of fixed level and varying weight, and show that this generating function is a rational function, whose coefficients belong to \mathbf{Q} in many cases of interest. We start with the case of full level, i.e., of modular forms on the upper half plane that are invariant (under the slash operator $f|_\alpha$) with respect to $SL(2, \mathbf{Z})$. In that setting, the graded ring \mathcal{R} of modular forms on $SL(2, \mathbf{Z})$ is generated by the Eisenstein series E_4 and E_6 of weights 4 and 6, and we restrict ourselves for simplicity to the Hecke operator T_N with N prime. Our generating function $F_N(a, b, A, B)$ is built up from the coefficients obtained when we express each $T_N(E_4^i E_6^j)$ as a polynomial in E_4 and E_6 . (Actually, to simplify the calculations, we work instead with multiples of E_4 and E_6 corresponding to the coefficients a and b in the equation $y^2 = x^3 + ax + b$ of an elliptic curve.) Our main result is then that F_N is a rational function, i.e., a ratio of polynomials in the variables a, b, A, B , with coefficients in \mathbf{Q} . The rationality of F_N generalizes to Hecke operators on modular forms of any level, and the coefficients in the rational functions belong to \mathbf{Q} in most cases of interest; in general, however, the coefficients can belong to a cyclotomic field.

The fact that F_N and its generalizations are rational functions holds in a rather general setting that extends beyond elliptic modular forms, provided we have a finitely generated \mathbf{C} -algebra of modular forms of fixed level and varying weight (i.e., varying type at infinity). In some sense, this generalizes the fact that the generating function F_1 for the “identity” Hecke operator T_1 is related to the Hilbert (or Poincaré) series of the graded algebra of modular forms of fixed level and varying weight. Our strategy is to express the action of T_N in general as a trace between two such graded algebras of modular forms, extending the usual trace of modular forms from a smaller congruence subgroup Γ' to a larger subgroup Γ , defined by a sum over cosets of $\Gamma' \backslash \Gamma$.

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It would be interesting to try using the generating function F_N and its analogs to tackle some conjectures about p -adic slopes of Hecke operators, such as for example Conjecture 1 of [BC05] about the 2-adic slopes of T_2 acting on cusp forms for $SL(2, \mathbf{Z})$. Several other authors have made computations of such slopes in the context of overconvergent modular forms, and the appearance of rational generating functions for Hecke operators has been noticed before [Smi04], but the rational generating functions there are computed *ad hoc* from a less conceptual perspective than our generalized Hilbert series of the graded algebra of modular forms.

Although we have not pursued the matter in this article, we point out that the rationality of F_N and its generalizations also implies the rationality of the generating function for the *traces* of T_N . Thus we recover the result in [FOP04] that states in a special case that $\sum_{\kappa \geq 0} \text{tr } T_N|_{\mathcal{M}_\kappa(SL(2, \mathbf{Z}))} X^\kappa$ is rational.

Beyond giving an existence proof for the rationality of F_N , we describe a framework that allows us to compute the rational function directly for arbitrary prime N when we work with full level $SL(2, \mathbf{Z})$. The calculations can be done without any reference to q -expansions, simply via calculations of isogenies [Vél71] from an elliptic curve $y^2 = x^3 + ax + b$ for transcendental a, b to its quotient by a cyclic subgroup of order N . Our approach allows us to express F_N as the trace of the inverse of a matrix with polynomial entries in a, b, A, B . Something similar can in principle be done for a fixed Hecke operator on arbitrary level, provided we can compute convenient generators and relations for the \mathbf{C} -algebra of all modular forms of that level. In any case, the computations involve fairly large numbers and matrices fairly quickly, and we have contented ourselves with describing the generating functions for T_2, T_3 , and T_5 on the full modular group $SL(2, \mathbf{Z})$.

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2. HECKE OPERATORS AND TRACES ON ALGEBRAS

Let $\Gamma \subset SL(2, \mathbf{Z})$ be a congruence subgroup. We write $\mathcal{M}_\kappa(\Gamma)$ for the space of weight κ modular forms which are invariant with respect to Γ . Viewing the forms as holomorphic functions on the upper half plane \mathcal{H} , we can multiply two modular forms of weights κ and κ' to get a product of weight $\kappa + \kappa'$. Thus we can combine forms of all weights with respect to a fixed group Γ into a graded ring of modular forms

$$(2.1) \quad \mathcal{R}_\Gamma = \bigoplus_{\kappa \geq 0} \mathcal{M}_\kappa(\Gamma).$$

We abbreviate $\mathcal{R} = \mathcal{R}_{SL(2, \mathbf{Z})}$.

We use the usual notation $\Gamma_0(N), \Gamma_1(N)$, and $\Gamma(N)$ for the congruence subgroups of $\Gamma(1) = SL(2, \mathbf{Z})$. We know that the ring \mathcal{R} of modular forms on $\Gamma(1)$ is generated by the Eisenstein series of weights 4 and 6, which we multiply below by suitable

constants. We thus obtain modular forms $a(\tau)$ and $b(\tau)$, for $\tau \in \mathcal{H}$, such that

$$(2.2) \quad \begin{aligned} \mathcal{R} &= \mathcal{R}_{\Gamma(1)} = \mathbf{C}[a, b], \\ a &= a(\tau) = -\frac{15}{\pi^4} \sum_{0 \neq \ell \in L_\tau} \ell^{-4} = -\frac{1}{3} \left(1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n \right), \\ b &= b(\tau) = \frac{35}{\pi^6} \sum_{0 \neq \ell \in L_\tau} \ell^{-6} = \frac{2}{27} \left(1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n \right). \end{aligned}$$

Here $L_\tau = \mathbf{Z} + \mathbf{Z}\tau$ is the lattice in \mathbf{C} with basis $\{1, \tau\}$, and $q = \exp(2\pi i\tau)$, as usual. We have chosen the above normalization to simplify the equation for the elliptic curve $E_\tau = \mathbf{C}/L_\tau$, as well as the analytic isomorphism between \mathbf{C}/L_τ and E_τ :

$$(2.3) \quad \begin{aligned} E_\tau : y^2 &= x^3 + ax + b, \\ P = z + L_\tau \in \mathbf{C}/L_\tau &\mapsto (x_P, y_P) = \left(-\frac{\wp(z; L_\tau)}{\pi^2}, \frac{i\wp'(z; L_\tau)}{2\pi^3} \right) \in E_\tau. \end{aligned}$$

In the above normalization, the invariant differential $\omega = dx/y$ on E_τ corresponds to $2\pi i dz$, where z is the ‘‘coordinate’’ on \mathbf{C}/L_τ . We shall interchangeably consider modular forms as

- functions on \mathcal{H} ;
- functions of the triple $(E, \omega, \text{level structure})$ for an elliptic curve E and of an invariant differential ω on E ;
- functions on pairs $(L, \text{level structure})$ where $L \subset \mathbf{C}$ is a lattice.

We will omit mention of the level structure whenever this simplifies our notation. Thus $f(\tau) = f(E_\tau, \omega) = f(L_\tau)$ in our three ways to view a modular form.

To simplify our treatment, we only consider the Hecke operator T_N when N is prime. We use the usual normalization that is convenient for q -expansions,

$$(2.4) \quad f \in \mathcal{M}_\kappa(\Gamma(1)) \implies T_N f(\tau) = \frac{1}{N} \left(N^\kappa f(N\tau) + \sum_{t=0}^{N-1} f((\tau + t)/N) \right).$$

In terms of elliptic curves or lattices, this means that

$$(2.5) \quad \begin{aligned} T_N f(E, \omega) &= \frac{1}{N} \sum_{N\text{-isogenies } \pi: E \rightarrow E' \text{ with } \pi^*(\omega') = \omega} f(E', \omega'), \\ T_N f(L_\tau) &= \frac{1}{N} \sum_{L \subset L' \text{ with } [L':L] = N} f(L'). \end{aligned}$$

We extend the Hecke operator T_N additively, so that we have a \mathbf{C} -linear map $T_N : \mathcal{R} \rightarrow \mathcal{R}$. We also define a linear map (depending on N), sending $f \in \mathcal{R}$ to $f' \in \mathcal{R}_{\Gamma_0(N)}$, by additively extending the definition

$$(2.6) \quad f \in \mathcal{M}_\kappa(\Gamma(1)) \implies f'(\tau) = N^\kappa f(N\tau).$$

If Γ' and Γ are congruence subgroups with $\Gamma' \subset \Gamma$, then we define a trace map $\text{tr}_\Gamma^{\Gamma'} : \mathcal{M}_*(\Gamma') \rightarrow \mathcal{M}_*(\Gamma)$ by additively extending

$$(2.7) \quad f \in \mathcal{M}_\kappa(\Gamma') \implies \text{tr}_\Gamma^{\Gamma'} f = \sum_{\alpha \in \Gamma' \backslash \Gamma} f|_\kappa \alpha.$$

Thus α ranges over a finite set of coset representatives giving a disjoint union $\Gamma = \sqcup_{\alpha} \Gamma' \alpha$. As usual, $(f|_{\kappa} \alpha)(\tau) = f(\alpha\tau)j(\alpha, \tau)^{-\kappa}$. We immediately obtain the following result:

Proposition 2.1. (1) *The map $f \in \mathcal{R} \mapsto f' \in \mathcal{R}_{\Gamma_0(N)}$ is a homomorphism of algebras.*

(2) *The map $T_N : \mathcal{R} \rightarrow \mathcal{R}$ is given by*

$$(2.8) \quad T_N f = \frac{1}{N} \operatorname{tr}_{\Gamma(1)}^{\Gamma_0(N)} f',$$

for the trace map $\operatorname{tr}_{\Gamma(1)}^{\Gamma_0(N)} : \mathcal{R}_{\Gamma_0(N)} \rightarrow \mathcal{R}$.

The ‘‘matrix’’ of T_N with respect to the basis $\{a^i b^j \mid i, j \geq 0\}$ of \mathcal{R} is described by coefficients c_{ijkl}^N , for $i, j, k, \ell \geq 0$, where

$$(2.9) \quad T_N(a^i b^j) = \sum_{k, \ell \geq 0} c_{ijkl}^N a^k b^{\ell}.$$

(By looking at weights of forms, we see that $c_{ijkl}^N = 0$ unless $4i + 6j = 4k + 6\ell$, so the above sum is finite.) We combine these coefficients c_{ijkl}^N into a power series, thereby obtaining the following generating function:

$$(2.10) \quad \begin{aligned} F_N(a, b, A, B) &= \sum_{i, j, k, \ell \geq 0} c_{ijkl}^N A^i B^j a^k b^{\ell} \\ &= \sum_{i, j \geq 0} A^i B^j T_N(a^i b^j) \in \mathcal{R}[[A, B]] = \mathbf{C}[a, b][[A, B]]. \end{aligned}$$

Here A, B are formal variables, and we can view a, b as independent transcendental variables as well.

3. THE CASE $N = 2$

In this section, we have $\Gamma_0(2) = \Gamma_1(2)$, and the level structure that this parametrizes on a given elliptic curve E is a 2-torsion point $(e, 0) \in E$, corresponding to $P = 1/2 + L_{\tau} \in \mathbf{C}/L_{\tau}$. We can view $e = x_P$ as the weight 2 Eisenstein series

$$(3.1) \quad e(\tau) = -\pi^{-2} \wp(1/2; L_{\tau}) = -\frac{2}{3} \left[1 + 24 \sum_{n \geq 1} \left(\sum_{\text{odd } d|n} d \right) q^n \right] \in \mathcal{M}_2(\Gamma_1(2)).$$

(The above identity follows from the Fourier expansion of $\wp(z; L_{\tau})$ in terms of q and $\exp(2\pi iz)$.) We of course have the identity $e^3 + ae + b = 0$ in $\mathcal{R}_{\Gamma_0(2)} = \mathcal{R}_{\Gamma_1(2)}$. It follows that the two forms a and e are algebraically independent over \mathbf{C} , since otherwise a and b would be algebraically dependent.

Proposition 3.1. *The algebra $\mathcal{R}_{\Gamma_0(2)}$ is generated by a, b , and e in weights 4, 6, and 2, respectively, subject only to the relation $e^3 + ae + b = 0$. Thus we have*

$$(3.2) \quad \mathcal{R}_{\Gamma_0(2)} = \mathbf{C}[a, b, e]/(e^3 + ae + b) = \mathcal{R}[e]/(e^3 + ae + b),$$

and hence $\mathcal{R}_{\Gamma_0(2)}$ is a free \mathcal{R} -module of rank 3, with basis $\{1, e, e^2\}$.

Proof. We sketch a proof of this standard result. We have inclusions of graded rings $\mathcal{R} = \mathbf{C}[a, b] \subset \mathbf{C}[a, b, e]/(e^3 + ae + b) \subset \mathcal{R}_{\Gamma_0(2)}$. The Hilbert series (with respect to the weight) of $\mathcal{R}_{\Gamma_0(2)}$ is $1/(1-t^2)(1-t^4) = (1+t^2+t^4)/(1-t^4)(1-t^6)$ by standard formulas for the dimension $\dim \mathcal{M}_{\kappa}(\Gamma_0(2))$. On the other hand, the

subring $\mathbf{C}[a, b, e]/(e^3 + ae + b)$ already has the same Hilbert series as all of $\mathcal{R}_{\Gamma_0(2)}$, so they must be equal. Note that since $b = -e^3 - ae$, we could have phrased our result as $\mathcal{R}_{\Gamma_0(2)} = \mathbf{C}[a, e]$; however, we primarily wish to view $\mathcal{R}_{\Gamma_0(2)}$ as an \mathcal{R} -module, for the purpose of computing traces. \square

We can now represent elements of $\mathcal{R}_{\Gamma_0(2)}$ as 3×3 matrices with elements in \mathcal{R} by the regular representation of $\mathcal{R}_{\Gamma_0(2)}$ as an \mathcal{R} -algebra, with respect to, say, the basis $\{1, e, e^2\}$. We thus obtain:

Corollary 3.2. *There exists a unique homomorphism of \mathbf{C} -algebras $\varrho: \mathcal{R}_{\Gamma_0(2)} \rightarrow M_{3 \times 3}(\mathcal{R})$ such that*

$$(3.3) \quad \varrho(a) = \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix}, \quad \varrho(b) = \begin{pmatrix} b & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & b \end{pmatrix}, \quad \varrho(e) = \begin{pmatrix} 0 & 0 & -b \\ 1 & 0 & -a \\ 0 & 1 & 0 \end{pmatrix}.$$

The trace map from $\mathcal{R}_{\Gamma_0(2)}$ to \mathcal{R} can be computed as the matrix trace:

$$(3.4) \quad f \in \mathcal{R}_{\Gamma_0(2)} \implies \mathrm{tr}_{\Gamma_0(2)}^{\Gamma_0(2)} f = \mathrm{tr} \varrho(f).$$

Proof. We introduce the fields of fractions \mathcal{K} , $\mathcal{K}_{\Gamma_0(2)}$, and $\mathcal{K}_{\Gamma(2)}$ of the integral domains \mathcal{R} , $\mathcal{R}_{\Gamma_0(2)}$, and $\mathcal{R}_{\Gamma(2)}$; since we have taken fields of fractions without respecting the graded structure, these fields are less natural than function fields of modular curves or than graded rings of *meromorphic* modular forms (obtained by inverting only the nonzero homogeneous elements). The action $f \mapsto f|\alpha$ gives the integral domain $\mathcal{R}_{\Gamma(2)}$ an action of the finite group $G = \Gamma(1)/\Gamma(2) \cong SL(2, \mathbf{Z}/2\mathbf{Z})$, such that \mathcal{R} is the subring invariant under G , and $\mathcal{R}_{\Gamma_0(2)}$ is the subring invariant under $H = \Gamma_0(2)/\Gamma(2) \subset G$. This situation is mirrored in the fields of fractions, the main point being that elements of $\mathcal{K}_{\Gamma(2)}$ can be written in the form f/s with a G -invariant denominator $s \in \mathcal{R}$. (Given an arbitrary denominator, multiply above and below by the “conjugates” of the denominator.) This last observation, combined with Proposition 3.1, also shows that $\{1, e, e^2\}$ is a basis for $\mathcal{K}_{\Gamma_0(2)}$ as a vector space over \mathcal{K} . Elementary Galois theory applied to \mathcal{K} , $\mathcal{K}_{\Gamma_0(2)}$, and $\mathcal{K}_{\Gamma(2)}$ now yields our statement that the trace of the regular representation is equal to the sum over conjugates given by the action of representatives α for $H \backslash G \cong \Gamma_0(2) \backslash \Gamma(1)$. \square

Proposition 3.3. *The ring homomorphism $f \in \mathcal{R} \mapsto f' \in \mathcal{R}_{\Gamma_0(2)}$ of (2.6) satisfies*

$$(3.5) \quad a' = -4a - 15e^2, \quad b' = 22b + 14ae.$$

Proof. We know that $a' \in \mathcal{M}_4(\Gamma_0(2)) = \mathbf{C}a + \mathbf{C}e^2$ and $b' \in \mathcal{M}_6(\Gamma_0(2)) = \mathbf{C}b + \mathbf{C}ae$. Thus we must find constants μ_1, \dots, μ_4 such that $a' = \mu_1 a + \mu_2 e^2$ and $b' = \mu_3 b + \mu_4 ae$. We can do this in one of three ways: the first way is to compare q -expansions, the second way is to use the known traces of the Hecke operator T_2 on the spaces $\mathcal{M}_\kappa(\Gamma(1))$ for $\kappa \leq 12$, and the third way is to use Vélú’s formulas for isogenies, as described in Section 4 below. \square

Theorem 3.4. *The generating function $F_2(a, b, A, B)$ of (2.10) is*

$$(3.6) \quad F_2(a, b, A, B) = \frac{1}{2} \mathrm{tr} \left[(I - A\varrho(a'))(I - B\varrho(b')) \right]^{-1} = \frac{1}{2} \mathrm{tr} M^{-1},$$

where I is the 3×3 identity matrix, and M is the product

$$(3.7) \quad M = \begin{pmatrix} 1 + 4Aa & -15Ab & 0 \\ 0 & 1 - 11Aa & -15Ab \\ 15A & 0 & 1 - 11Aa \end{pmatrix} \begin{pmatrix} 1 - 22Bb & 0 & 14Bab \\ -14Ba & 1 - 22Bb & 14Ba^2 \\ 0 & -14Ba & 1 - 22Bb \end{pmatrix}.$$

Proof. We extend the operations $f \mapsto f'$ and $\text{tr}_{\Gamma(1)}^{\Gamma_0(2)}$ coefficientwise, so that they map from $\mathcal{R}[[A, B]]$ to $\mathcal{R}_{\Gamma_0(2)}[[A, B]]$ and vice-versa. We similarly extend the regular representation ϱ so that it sends $\mathcal{R}_{\Gamma_0(2)}[[A, B]]$ to $M_{3 \times 3}(\mathcal{R}[[A, B]])$. All relations are still valid after this extension of scalars; in particular $\text{tr}_{\Gamma(1)}^{\Gamma_0(2)}$ can again be computed as a matrix trace. Now by the results of Section 2, and by linearity in each coefficient of a monomial $A^i B^j$, we have

$$(3.8) \quad F_2(a, b, A, B) = \frac{1}{2} \text{tr}_{\Gamma(1)}^{\Gamma_0(2)} \left[\sum_{i, j \geq 0} A^i B^j (a')^i (b')^j \right] = \frac{1}{2} \text{tr}_{\Gamma(1)}^{\Gamma_0(2)} \left[\frac{1}{(1 - Aa')(1 - Bb')} \right].$$

We know the values for a', b' from (3.5). We then compute the trace as in (3.4); note that the regular representation ϱ respects inverses in the power series ring $\mathcal{R}[[A, B]]$. \square

Note that it is clear from the above that F_2 is a rational function in $\mathbf{Q}(a, b, A, B)$. Moreover, the denominator of F_2 is the determinant $\det M$, which appears in the denominator of M^{-1} .

4. GENERAL FRAMEWORK FOR T_N IN TERMS OF VÉLU'S FORMULAS

In this section, N is an odd prime. One can prove a version of the results below even if $N = 2$, or indeed if N is composite, but the statements and proofs become more complicated.

By the same reasoning as in Theorem 3.4, we deduce that

$$(4.1) \quad F_N(a, b, A, B) = \frac{1}{N} \text{tr}_{\Gamma(1)}^{\Gamma_0(N)} \left[\frac{1}{(1 - Aa')(1 - Bb')} \right].$$

At this point, $a', b' \in \mathcal{R}_{\Gamma_0(N)}$, and we wish to find their values and be able to represent them as matrices via a suitable generalization of the regular representation ϱ from Section 3. It is enough to work on the level of the fields of fractions $\mathcal{K}, \mathcal{K}_{\Gamma_0(N)}$ introduced in the proof of Corollary 3.2, and to calculate with the regular representation with respect to a \mathcal{K} -basis of $\mathcal{K}_{\Gamma_0(N)}$. We introduce a further simplification by passing to the smaller congruence subgroup

$$(4.2) \quad \Gamma_{\pm}(N) = \{\pm I\} \cdot \Gamma_1(N) \subset \Gamma_0(N),$$

for which, as we shall see, the corresponding field of fractions $\mathcal{K}_{\Gamma_{\pm}(N)}$ is easier to describe and to study via a regular representation than the original $\mathcal{K}_{\Gamma_0(N)}$. Thus we can easily evaluate the desired trace with respect to this smaller group. The resulting trace is off by a factor $[\Gamma_{\pm}(N) : \Gamma_0(N)] = (N - 1)/2$, so we retrieve the original trace by dividing:

$$(4.3) \quad \text{tr}_{\Gamma(1)}^{\Gamma_0(N)} [(1 - Aa')(1 - Bb')]^{-1} = \frac{2}{N - 1} \text{tr}_{\Gamma(1)}^{\Gamma_{\pm}(N)} [(1 - Aa')(1 - Bb')]^{-1}.$$

In order to describe $\mathcal{K}_{\Gamma_{\pm}(N)}$, we begin with the level structure parametrized by $\Gamma_1(N)$, namely the N -torsion point on our varying elliptic curve, corresponding to

$P = 1/N + L_\tau \in \mathbf{C}/L_\tau$. Since $\Gamma_\pm(N)$ introduces an ambiguity between P and $-P$, the x -coordinate x_P of P is invariant under $\Gamma_\pm(N)$. Hence we obtain

$$(4.4) \quad x_P(\tau) = -\pi^{-2} \wp(1/N; L_\tau) \in \mathcal{M}_2(\Gamma_\pm(N)),$$

and, more generally,

$$(4.5) \quad x_{\ell P}(\tau) = -\pi^{-2} \wp(\ell/N; L_\tau) \in \mathcal{M}_2(\Gamma_\pm(N)), \quad \ell \in S = \{1, \dots, (N-1)/2\}.$$

(The above modular forms are, incidentally, all Eisenstein series.) We do not need other values of ℓ , since $x_{-\ell P} = x_{\ell P}$. We begin with the following proposition:

Proposition 4.1. *Let $\psi_N(x; a, b) = Nx^{(N^2-1)/2} + \dots \in \mathbf{Z}[a, b, x]$ be the N -division polynomial (see, e.g., Exercise III.3.7 of [Sil86]).*

- (1) *We have $\mathcal{K}_{\Gamma_\pm(N)} = \mathcal{K}[x_P]/(\psi_N(x_P; a, b))$.*
- (2) *The powers $\{1, x_P, x_P^2, \dots, x_P^{(N^2-3)/2}\}$ are a basis for $\mathcal{K}_{\Gamma_\pm(N)}$ over \mathcal{K} .*
- (3) *The other modular forms $x_{\ell P}$ for $\ell \in S$ also belong to $\mathcal{K}_{\Gamma_\pm(N)}$; their expressions in terms of x_P , a , and b are straightforward to compute, and involve only coefficients from \mathbf{Q} .*
- (4) *Every symmetric polynomial in the $x_{\ell P}$ belongs to $\mathcal{R}_{\Gamma_0(N)}$.*

Proof. By Galois theory for the extension $\mathcal{K}_{\Gamma(N)}/\mathcal{K}$, we know that x_P generates $\mathcal{K}_{\Gamma_\pm(N)}$ over \mathcal{K} , because x_P is left invariant precisely by the subgroup $\Gamma_\pm(N)$ of $\Gamma(1)$. Moreover, x_P is a root of the division polynomial $\psi_N(x; a, b)$, which is irreducible over \mathcal{K} because $\Gamma(1)$ acts transitively on the (nonzero) torsion points in $E[N]$, and hence on the roots of ψ_N . (Note that if N were not prime, we would need to work with the ‘‘primitive’’ N -division polynomial instead of ψ_N .) This proves the first two statements of the proposition. The third statement holds from the multiplication formula $x_{\ell P} = \phi_\ell(x_P; a, b)/[\psi_\ell(x_P; a, b)]^2$, where ψ_ℓ is the ℓ -division polynomial, and $\phi_\ell \in \mathbf{Z}[a, b, x]$ is another polynomial that is straightforward to compute. The fourth statement is immediate, since the action of an element of $\Gamma_0(N)$ transforms P into a multiple $\ell'P$ with $(\ell', N) = 1$. \square

In light of the above discussion, it is straightforward to calculate F_N provided we can express a' and b' as elements of $\mathcal{K}[x_P]/(\psi_N(x_P; a, b))$. This is not as easily done as in the case $N = 2$, where we simply compared q -expansions, since the expressions that we seek for a' and b' are rational functions and not necessarily polynomials in x_P , a , and b . Thus, if we wished to use q -expansions in order to find expressions for a' , b' , we would need to bound the denominators of those expressions. We instead compute a' , b' using the interpretation of modular forms as functions of elliptic curves E with a choice of global differential ω .

Proposition 4.2. *We have*

$$(4.6) \quad \begin{aligned} a' &= a - 30 \sum_{\ell \in S} x_{\ell P}^2 - 5(N-1)a, \\ b' &= b - 70 \sum_{\ell \in S} x_{\ell P}^3 - 42a \sum_{\ell \in S} x_{\ell P} - 14(N-1)b. \end{aligned}$$

Proof. According to Vélú’s formulas for isogenies [Vél71], the coefficients a' , b' above are the coefficients in a Weierstrass equation $E' : y'^2 = x'^3 + a'x' + b'$ for the quotient curve $E' = E/\langle x_P \rangle$ of the elliptic curve $E : y^2 = x^3 + ax + b$ by the subgroup generated by $P \in E[N]$. (Recall that N is odd; Vélú’s formulas are slightly different

when 2-torsion is involved.) The isogeny given by the projection $\pi : E \rightarrow E'$ is such that the pullback of the global differential $\omega' = dx'/y'$ is $\pi^*\omega' = \omega = dx/y$. Thus a' (respectively, b') corresponds to one term in the sum over isogenies defining $T_N a$ (respectively, $T_N b$) in (2.5). In fact, a' specifically corresponds to the term $N^k a(N\tau)$ in (2.4), and similarly for b' . The reason is that if we view E as \mathbf{C}/L_τ , ω as $2\pi i dz$, and P as the image of $1/N$, then E' is $\mathbf{C}/(\mathbf{Z}\frac{1}{N} + \mathbf{Z}\tau)$ with $\omega' = 2\pi i dz$ as well; hence $a'(\tau) = a(\mathbf{Z}\frac{1}{N} + \mathbf{Z}\tau) = N^4 a(L_{N\tau})$, in the interpretation of $a = a(L)$ as a function of lattices. \square

Putting together the results of this section, we obtain the main result of this article:

Theorem 4.3. *The generating function F_N defined in (2.10) is a rational function in $\mathbf{Q}(a, b, A, B)$, and F_N can be explicitly computed for any specific value of N .*

Proof. The case $N = 2$ is Theorem 3.4. For $N \geq 3$, combine equations (4.1) and (4.3), as well as Propositions 4.1 and 4.2. The trace $\text{tr}_{\Gamma_{\pm(N)}}^{\Gamma_{\pm(N)}}$ in (4.3) can be computed via the regular representation of $\mathcal{K}_{\Gamma_{\pm(N)}}$ as a \mathcal{K} -algebra, with respect to the basis consisting of the powers of x_P . All the constants that we encounter belong to \mathbf{Q} , most significantly by the third statement in Proposition 4.1. \square

We remark that it is unfortunate that we need to use square matrices of size $(N^2 - 1)/2 = [\Gamma(1) : \Gamma_{\pm(N)}]$ in our regular representation of $\mathcal{K}_{\Gamma_{\pm(N)}}$, since the true dimension that matters is $N + 1 = [\Gamma(1) : \Gamma_0(N)]$. It would be agreeable to have a direct way to describe a cyclic N -subgroup of E and the isogeny obtained by quotienting E by that subgroup. This would be simpler than our approach of choosing the N -torsion point P and considering the subgroup $\langle P \rangle$ generated by P .

5. CALCULATIONS FOR T_3 AND T_5

In this section, we give explicit matrices $\varrho(a')$ and $\varrho(b')$, in the two cases $N = 3$ and $N = 5$, for a suitable regular representation ϱ of $\mathcal{K}_{\Gamma_0(N)}$. (Note that $\Gamma_0(3) = \Gamma_{\pm}(3)$, but $\Gamma_0(5) \neq \Gamma_{\pm}(5)$, so we must modify the approach in Section 4 when $N = 5$.) This is enough data to describe F_3 and F_5 completely, by evaluating the trace in (4.1) using ϱ .

As noted in the above paragraph, the case $N = 3$ is exactly covered by our previous methods. In this case $\psi_3 = 3x^4 + 6ax^2 + 12bx - a^2$, so $\varrho(a) = aI$ and $\varrho(b) = bI$, where I is the 4×4 identity matrix, and

$$(5.1) \quad \varrho(x_P) = \begin{pmatrix} 0 & 0 & 0 & a^2/3 \\ 1 & 0 & 0 & -4b \\ 0 & 1 & 0 & -2a \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

We fortunately have $S = \{1\}$, so $a' = -9a - 30x_P^2$ and $b' = -27b - 70x_P^3 - 42ax_P$. Hence we obtain the following result.

Theorem 5.1. *For $N = 3$, we have that $\varrho(a')$ and $\varrho(b')$ are respectively*

$$(5.2) \quad \begin{pmatrix} -9a & 0 & -10a^2 & 0 \\ 0 & -9a & 120b & -10a^2 \\ -30 & 0 & 51a & 120b \\ 0 & -30 & 0 & 51a \end{pmatrix} \text{ and } \begin{pmatrix} -27b & -70a^2/3 & 0 & 98a^3/3 \\ -42a & 253b & -70a^2/3 & -392ab \\ 0 & 98a & 253b & -658a^2/3 \\ -70 & 0 & 98a & 253b \end{pmatrix}.$$

We now turn to the case $N = 5$. In this setting, we first worked as described in Section 4, using the regular representation of $\mathcal{K}_{\Gamma_{\pm}(5)}$ with respect to the basis $\{1, x_P, x_P^2, \dots, x_P^{11}\}$. We were however dissatisfied with the appearance of the results — our first calculations gave us 12×12 matrices $\varrho(a')$, $\varrho(b')$ whose entries all had a denominator of $(4a^3 + 27b^2)^2$, i.e., the square of the discriminant. We preferred instead to work with elements invariant under $\Gamma_0(5)$, and so looked for “nice” symmetric polynomials in x_P, x_{2P} (corresponding to $S = \{1, 2\}$) that gave a basis for the 6-dimensional field extension $\mathcal{K}_{\Gamma_0(5)}/\mathcal{K}$. After some trial and error, we settled on the following ordered basis, which gave us matrices with polynomial entries:

$$(5.3) \quad \{f_1, \dots, f_6\} = \{1, x_P + x_{2P}, x_P x_{2P}, x_P^2 + x_{2P}^2, x_P^2 x_{2P}^2, x_P^3 + x_{2P}^3\}.$$

Remark 5.2. To help the reader check the calculations, we mention that in the regular representation ϱ with respect to the above basis $\{f_1, \dots, f_6\}$, we have

$$(5.4) \quad \varrho(x_P + x_{2P}) = \begin{pmatrix} 0 & 0 & -4b/3 & -4b/3 & 16ab/3 & 3a^2 \\ 1 & 0 & -2a/3 & -2a/3 & 11a^2/3 & -8b \\ 0 & 2 & 0 & 0 & -12b & -2a \\ 0 & 1 & 0 & 0 & -4b & 2a \\ 0 & 0 & 0 & 0 & 0 & 15 \\ 0 & 0 & 1/3 & 4/3 & -4a/3 & 0 \end{pmatrix},$$

$$(5.5) \quad \varrho(x_P x_{2P}) = \begin{pmatrix} 0 & -4b/3 & 0 & a^2 & (16b^2 - 4a^3)/5 & 40ab/3 \\ 0 & -2a/3 & 0 & -4b & 24ab/5 & 29a^2/3 \\ 1 & 0 & 0 & -2a & 9a^2/5 & -32b \\ 0 & 0 & 0 & 0 & a^2/5 & -12b \\ 0 & 0 & 1 & 3 & -18a/5 & 0 \\ 0 & 1/3 & 0 & 0 & -8b/5 & -10a/3 \end{pmatrix}.$$

This can be checked by working in the full field $\mathcal{K}_{\Gamma_{\pm}(5)}$ in terms of the basis of powers of x_P . Alternatively, the reader may wish to verify the above matrices by working directly from the algebraic relations satisfied by x_P and x_{2P} over \mathcal{K} . Instead of using the cumbersome fact that x_P and x_{2P} are roots of the high degree division polynomial ψ_5 , it is easier to note that $x_{4P} = x_P$ and to use the duplication formula for points on E to deduce the relations $[\psi_2(x_P)]^2 x_{2P} = \phi_2(x_P)$ and $[\psi_2(x_{2P})]^2 x_P = \phi_2(x_{2P})$. (We also need the fact that $x_P \neq x_{2P}$ since P does not have order 3; the ideal of relations between x_P and x_{2P} can be obtained by starting with the ideal generated by the two formulas above and by saturating that ideal with respect to $x_P - x_{2P}$.)

Using (5.4) and (5.5), we now easily find $\varrho(x_P^2 + x_{2P}^2)$ and $\varrho(x_P^3 + x_{2P}^3)$, which allow us to apply Vélú’s formulas to obtain the following result:

Theorem 5.3. *In the case $N = 5$, let ϱ be the regular representation of $\mathcal{K}_{\Gamma_0(5)}$ over \mathcal{K} with respect to the basis $\{f_1, \dots, f_6\}$ above. Then we have*

$$(5.6) \quad \varrho(a') = \begin{pmatrix} -19a & 40b & -30a^2 & -60a^2 & 72a^3 - 448b^2 & -1600ab \\ 0 & a & 120b & 120b & -512ab & -1160a^2 \\ 0 & 0 & 41a & 0 & -192a^2 & 3960b \\ -30 & 0 & 0 & -79a & -18a^2 & 1320b \\ 0 & 0 & -90 & -420 & 365a & 0 \\ 0 & -40 & 0 & 0 & 184b & 321a \end{pmatrix},$$

$$(5.7) \quad \varrho(b') = \begin{pmatrix} -55b & -210a^2 & -2632ab/3 & -11032ab/3 & 13888a^2b/3 & 1554a^3 - 12320b^2 \\ -42a & 505b & -1946a^2/3 & -8036a^2/3 & 7658a^3/3 - 3360b^2 & -13104ab \\ 0 & 56a & 2185b & 9240b & -9576ab & -5096a^2 \\ 0 & -182a & 840b & 3025b & -2632ab & -994a^2 \\ 0 & -1050 & 0 & 0 & 4705b & 7770a \\ -70 & 0 & 658a/3 & 2212a/3 & -2842a^2/3 & 5265b \end{pmatrix}.$$

6. GENERALIZATIONS

6.1. Other levels than $\Gamma(1)$. Our first generalization is to study generating functions for Hecke operators on \mathcal{R}_Γ , for an arbitrary congruence subgroup Γ . Our approach can deal with any Hecke operator given by a double coset $\Gamma\alpha\Gamma$ with $\alpha \in GL(2, \mathbf{Q})$, $\det \alpha > 0$. In this situation, we shall show in this subsection that the analog of F_N is still a rational function, with coefficients in a number field; in many cases of interest, the coefficients actually lie in \mathbf{Q} . We can compute the analog of F_N in any specific case, but we do not have a satisfactory systematic method for computing the generating function by methods analogous to those in Section 4.

The main issue in generalizing our previous argument to arbitrary Γ is that as soon as the modular curve associated to Γ has positive genus, the ring \mathcal{R}_Γ is no longer a polynomial algebra in two variables. Hence the \mathbf{C} -basis $\{a^i b^j \mid i, j \geq 0\}$ of $\mathcal{R} = \mathcal{R}_{\Gamma(1)}$ that we used to define the coefficients c_{ijkl}^N of (2.9) must be replaced by something more complicated for \mathcal{R}_Γ . We must do this in a way that still yields an analog of the identity $\sum_{i,j \geq 0} A^i B^j a^i b^j = [(1 - Aa)(1 - Bb)]^{-1}$ in $\mathcal{R}[[A, B]]$ that plays such a crucial role in Theorems 3.4 and 4.3. We thus replace $a, b \in \mathcal{R}$ by generators $a_1, \dots, a_r \in \mathcal{R}_\Gamma$, where $a_i \in \mathcal{M}_{\kappa_i}(\Gamma)$. (The ring \mathcal{R}_Γ is finitely generated as a \mathbf{C} -algebra because, e.g., it is an integral extension of \mathcal{R} .) Writing I for the ideal of relations among the $\{a_i\}$, we see that we need to find an appropriate \mathbf{C} -basis for $\mathcal{R}_\Gamma = \mathbf{C}[a_1, \dots, a_r]/I$.

Proposition 6.1. *Given any finitely generated \mathbf{C} -algebra $\mathbf{C}[a_1, \dots, a_r]/I$, let $\text{in}(I)$ be the initial ideal of I with respect to any fixed term order on the monomials in the $\{a_i\}$. Then the set*

$$(6.1) \quad \mathcal{B} = \{\text{monomials } m \mid m \notin \text{in}(I)\}$$

is a \mathbf{C} -basis for $\mathbf{C}[a_1, \dots, a_r]/I$, and the formal power series

$$(6.2) \quad G(a_1, \dots, a_r) = \sum_{m \in \mathcal{B}} m \in \mathbf{C}[[a_1, \dots, a_r]]$$

is actually a rational function, of the form

$$(6.3) \quad G(a_1, \dots, a_r) = \frac{N(a_1, \dots, a_r)}{(1 - a_1)(1 - a_2) \dots (1 - a_r)}$$

where $N(a_1, \dots, a_r) \in \mathbf{Z}[a_1, \dots, a_r]$ is a polynomial with integer coefficients.

Proof. The first assertion, that \mathcal{B} is a basis, is a standard result in the theory of Gröbner bases. The second assertion, that G is a rational function with known denominator, follows from a direct modification of the usual argument by induction on r to show the rationality of Hilbert series of graded modules. We apply this specifically to the module $M = \mathbf{C}[a_1, \dots, a_r]/\text{in}(I)$, which carries an action of the

algebraic torus $T = (\mathbf{C}^*)^r$ such that an element $t = (\lambda_1, \dots, \lambda_r) \in T$ sends a_i to $ta_i = \lambda_i a_i$, for $1 \leq i \leq r$. Thus the series G is the same as the T -equivariant Hilbert series of M discussed, e.g., in Section 6.6 of [CG97]. \square

Remark 6.2. The function G depends significantly on the choice of term order, to say nothing of the choice of generators a_1, \dots, a_r . Take for example $\mathcal{R}_{\Gamma_0(2)} = \mathbf{C}[a, b, e]/(e^3 + ae + b)$. Depending on whether the initial term in $e^3 + ae + b$ is e^3 , ae , or b , we obtain $G = (1 + e + e^2)/[(1-a)(1-b)]$, $G = (1 - ae)/[(1-a)(1-b)(1-e)]$, or $G = 1/[(1-a)(1-e)]$, respectively. These examples incidentally show us that cancellation can occur between the numerator and denominator of G .

In light of the above proposition, we now see that the analog of our earlier sum $\sum_{i,j} A^i B^j a^i b^j$ is

$$(6.4) \quad G(A_1 a_1, \dots, A_r a_r) = \sum_{\text{monomials } m=a_1^{i_1} \dots a_r^{i_r} \in \mathcal{B}} A_1^{i_1} \dots A_r^{i_r} a_1^{i_1} \dots a_r^{i_r},$$

while our analog of F_N , corresponding to the Hecke operator $\Gamma\alpha\Gamma$, is

$$(6.5) \quad F_{\Gamma\alpha\Gamma}(A_1, \dots, A_r, a_1, \dots, a_r) = \sum_{m=a_1^{i_1} \dots a_r^{i_r} \in \mathcal{B}} A_1^{i_1} \dots A_r^{i_r} \left[a_1^{i_1} \dots a_r^{i_r} \mid \Gamma\alpha\Gamma \right].$$

We view the formal sums above in the ring $\mathcal{R}_\Gamma[[A_1, \dots, A_r]]$. Our argument for the rationality of $F_{\Gamma\alpha\Gamma}$ now proceeds essentially identically to our previous discussion, and we obtain (up to normalization constants) an identity of the form

$$(6.6) \quad F_{\Gamma\alpha\Gamma} = \text{tr}_{\Gamma'}^{\Gamma} G(A_1 a'_1, \dots, A_r a'_r), \quad \text{where } \Gamma' = \Gamma \cap \alpha^{-1}\Gamma\alpha \text{ and } f' = f \mid \alpha.$$

More precisely, depending on how we want to normalize the action of $\Gamma\alpha\Gamma$, we can modify the definition of f' by choosing a constant C and defining $f \in \mathcal{M}_\kappa(\Gamma) \implies f' = C^\kappa f \mid_{\kappa} \alpha$; this ensures that the map $f \mapsto f'$ is still a ring homomorphism from \mathcal{R}_Γ to $\mathcal{R}_{\Gamma'}$. We can also include another constant factor in front of the trace in (6.6).

The above suffices to show that $F_{\Gamma\alpha\Gamma}$ is a rational function of the $\{a_i\}$ and the $\{A_i\}$, viewing these as independent indeterminates. The coefficients of this rational function can be taken to lie in a field containing essentially the coefficients of the q -expansions of all modular forms in \mathcal{R}_Γ and $\mathcal{R}_{\Gamma'}$ that we encounter.

Theorem 6.3. *Let L be a subfield of \mathbf{C} such that for all weights κ , the spaces $\mathcal{M}_\kappa(\Gamma)$ and $\mathcal{M}_\kappa(\Gamma')$ have a basis of forms whose q -expansions have L -rational coefficients, and such that if $f \in \mathcal{M}_\kappa(\Gamma)$ has L -rational coefficients, then so does f' . Assume that the generators $\{a_i\}$ of \mathcal{R}_Γ are moreover chosen to all have L -rational coefficients. Then the generating function $F_{\Gamma\alpha\Gamma}$ is a rational function in the indeterminates $a_1, \dots, a_r, A_1, \dots, A_r$, with coefficients belonging to L .*

Proof. The assumptions on the coefficients in the q -expansions allow us to choose a \mathcal{K}_Γ -basis for $\mathcal{K}_{\Gamma'}$, with respect to which the regular representation ϱ takes a form $f \in \mathcal{M}_\kappa(\Gamma')$ with L -rational coefficients to a matrix $\varrho(f)$ whose entries are rational functions of the $\{a_i\}$ with coefficients in L . (Note that the ideal I of relations between the $\{a_i\}$ is also defined over L .) Since the forms $\{a'_i\}$ are also L -rational by assumption, we obtain our desired result. \square

Corollary 6.4. *We can always take L above to be a cyclotomic field (provided the normalizing constants like C above also belong to L). In the typical case where Γ*

is one of $\Gamma_0(N)$, $\Gamma_1(N)$, or $\Gamma(N)$, and α is a diagonal matrix, then we can take $L = \mathbf{Q}$. This typical case includes the “standard” Hecke operators T_ℓ on any one of the groups above, even when $(\ell, N) > 1$.

Proof. It is well known that the space $\mathcal{M}_\kappa(\Gamma(N))$ (similarly for $\Gamma_0(N)$ and $\Gamma_1(N)$) has a basis of forms whose q -expansions have \mathbf{Q} -rational coefficients, but that replacing f by $f|_\kappa\alpha$ where α is an integral matrix with determinant ℓ can introduce roots of unity up to $N\ell$ (see for example Chapters 3 and 6 of [Shi71]). Thus the first assertion of our corollary is clear. The second assertion follows because the action of a diagonal matrix α replaces τ by a multiple $a\tau/d$ for some $a, d \in \mathbf{Z}$, thereby preserving rationality of q -expansions. \square

6.2. Hecke operators restricted to cusp forms. Our second generalization is that our method extends to construct rational generating functions for Hecke operators acting only on the cuspidal part $\mathcal{I}_\Gamma = \bigoplus_{\kappa > 0} \mathcal{S}_\kappa(\Gamma)$ of the full ring of modular forms \mathcal{R}_Γ . This requires very little work in light of our observations in Subsection 6.1. We merely need to point out that \mathcal{I}_Γ is an ideal in \mathcal{R}_Γ . Hence we can use a Gröbner basis argument as in Proposition 6.1 to produce a rational function, analogous to G , which is the formal sum of a basis for all cusp forms. It is probably best to choose the generators $\{a_i\}$ of the \mathbf{C} -algebra \mathcal{R}_Γ to consist of cusp forms and Eisenstein series, in such a way that \mathcal{I}_Γ is generated by $\{a_i \mid a_i \text{ is a cusp form}\}$.

6.3. Automorphic forms on other groups. We conclude with the observation that the results in this article generalize to other settings where one has graded rings of automorphic forms. This includes groups with Hermitian symmetric spaces, for which we can interpret the automorphic forms as holomorphic functions on domains in \mathbf{C}^n , such as the case of Hilbert modular forms (over a totally real number field) and Siegel modular forms. However, in the case of Hilbert modular forms, we would probably be restricted to parallel weights, in order to obtain a graded ring of automorphic forms that is a finitely generated \mathbf{C} -algebra. We do not need modular varieties with cusps to carry out our program; modular forms on indefinite quaternion algebras over \mathbf{Q} come to mind, corresponding to automorphic forms on Shimura curves, but it is more complicated to compute relations between the analogs of the forms a_i and a'_i in that setting.

An alternative source of graded rings of automorphic forms is groups G for which $G(\mathbf{R})$ is compact, as discussed in [KM01]. In that setting, one can view modular forms as holomorphic sections of line bundles on several disjoint copies of the complex flag variety associated to G . In this setting, we still have a ring $\mathcal{R}_\Gamma = \mathbf{C}[a_1, \dots, a_r]/I$. However, \mathcal{R}_Γ is no longer an integral domain, and we cannot work with fields of fractions analogous to \mathcal{K}_Γ ; this may cause some difficulties in generalizing the regular representation ϱ to this situation. At any rate, we do not need ϱ if our main goal is to prove that the generating functions are rational. We simply work with the trace from Γ' to Γ as defined by coset representatives; this works best if we pass to a smaller subgroup $\Gamma'' \subset \Gamma' \subset \Gamma$ such that Γ'' is a normal subgroup of Γ , similarly to taking $\Gamma'' = \Gamma(N)$ earlier in this article. We then do our computations in the algebra \mathcal{A} obtained as a localization of $\mathcal{R}_{\Gamma''}[A_1, \dots, A_r]$ by inverting all elements that are congruent to 1 modulo the ideal (A_1, \dots, A_r) . (These elements are already invertible in $\mathcal{R}_{\Gamma''}[[A_1, \dots, A_r]]$, and so \mathcal{A} injects into the power series ring.) Then the expression corresponding to $F = \text{tr}_{\Gamma'}^{\Gamma} G(A_1 a'_1, \dots, A_r a'_r)$ is

a Γ -invariant expression in \mathcal{A} , which can be put over a Γ -invariant common denominator (the “norm” of $\prod_i(1 - A_i a'_i)$). Then our generating function corresponding to F has a numerator and a denominator in $\mathcal{R}_{\Gamma''}[A_1, \dots, A_r]$ that are both invariant under Γ , since F itself is invariant. It follows that the coefficients of the numerator and denominator of F are Γ -invariant elements of $\mathcal{R}_{\Gamma''}$, i.e., elements of \mathcal{R}_{Γ} , as desired.

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