Laplace Transforms and Differential Equations

Introduction:

The Laplace transform is an integral transform that converts a function of time into a function of complex frequency.

Transforms differential operators into multiplication, so differential equations become algebraic equations.

Definition of the Laplace Transform:

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt.$$

Where s is a complex number and f(t) is a real function.

Other notations:

$$\mathcal{L}\{f\} \quad \text{or} \quad \mathcal{L}\{f(t)\}$$

Inverse Laplace Transform:

$$f(t) = \mathcal{L}^{-1}\{F\}(t) = \frac{1}{2\pi i} \lim_{T\to\infty} \int_{\gamma - iT}^{\gamma + iT} e^{st} F(s)\, ds,$$

Inverses:

Given the functions f(t) and g(t), and their respective Laplace transforms *F*(*s*) and *G*(*s*)

$$f(t) = \mathcal{L}^{-1}\{F(s)\},$$
$$g(t) = \mathcal{L}^{-1}\{G(s)\},$$

Linearity:

$$\mathcal{L}\{f(t) + g(t)\} = \mathcal{L}\{f(t)\} + \mathcal{L}\{g(t)\}$$
$$\mathcal{L}\{af(t)\} = a\mathcal{L}\{f(t)\}$$

General Differentiation:

$$s^n F(s) - \sum_{k=1}^{n} s^{n-k} f^{(k-1)}(0)$$

So, differentiating f(t) once , (f ' (t)), gives us:

$$sF(s) - f(0)$$

Example: Radioactive Decay

The rate of radioactive decay is given by the following first order differential equation:

$$\frac{dN}{dt} + \lambda N = 0.$$

Then we apply the Laplace transform to each element of the equation:

$$\left(s\tilde{N}(s) - N_o\right) + \lambda \tilde{N}(s) = 0,$$

Where

$$\tilde{N}(s) = \mathcal{L}\{N(t)\}$$

And

$$N_o = N(0).$$

Then solving,

$$\tilde{N}(s) = \frac{N_o}{s + \lambda}.$$

Then to find the general solution, apply the inverse Laplace transform.