The Poisson Integral Formula

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$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{\left(r_0^2 - r^2\right)u(r_0,\phi)}{r_0^2 - 2r_0 r cos(\phi - \theta) + r^2} d\phi \qquad (r < r_0)$$

- Motivation: The Heat Equation $\nabla^2 u = a^2 \frac{\partial u}{\partial t}$
  - Determines temperature on the inside of a domain *D* given information at the boundary (boundary conditions) ∂D.
  - Of interest to engineering and other real world situations. Many of which can be modeled using similar equations.
- Dirichlet Problem
  - Note how in the earlier heat equation, the boundary conditions were known and from these we wanted to derive the function on its domain.
  - The Dirichlet Problem specifically requires two things: one, that we have a function harmonic in some domain and two, the value of the function at the boundary of the domain.
  - Initially used for real world problems, eventually became a field in and of itself.
- Poisson Integral Formula and Dirichlet Problem
  - Here is the Dirichlet Problem for the domain of a disk.
  - $\nabla^2 u(r,\theta) = 0$
  - $u(r_0,\theta) = f(\theta)$
  - Poisson Integral Formula uniquely solves all problems with this form.

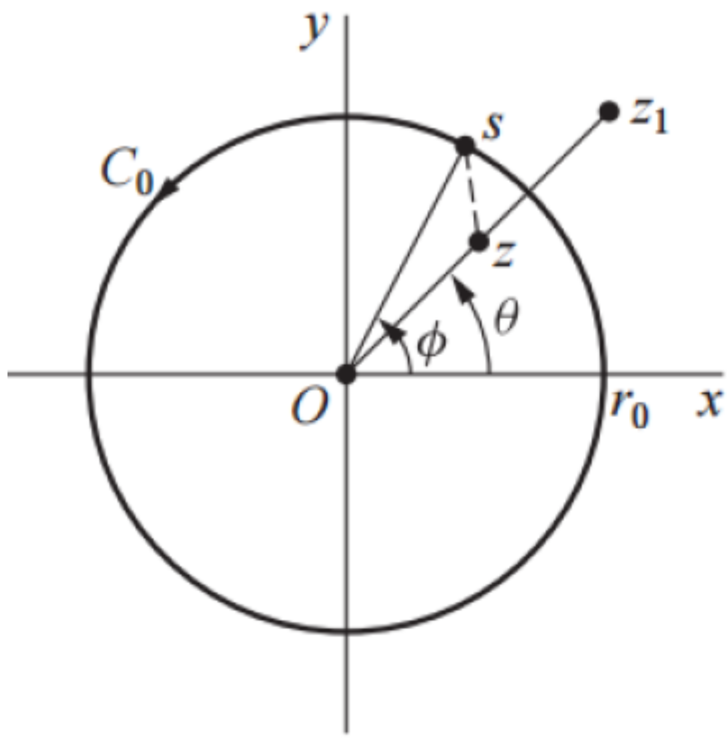

FIGURE 177

- Extra: If a solution exists, it must be unique by the Maximum Principle (for Dirichlet at least).

- Connections to Complex
  - Poisson Integral Formula can be derived from the familiar Cauchy Integral Formula
  - $f(z) = \frac{1}{2\pi i} \oint_{C_0} \frac{f(z)}{s-z} ds$
  - Derivation involves concept of the "inverse with respect to the circle" as well as extracting the real part from a complex formula.