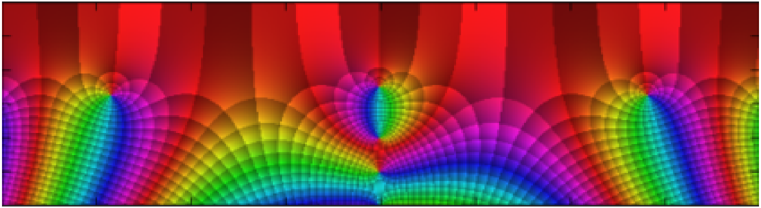

# Riemann Zeta Function

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# What is the $\zeta$ function?

Consider the series

$$Z(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

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**Example**

$$Z(1) = 1 + \frac{1}{2} + \frac{1}{3} + \dots$$

This is the harmonic series, which we know diverges.

# What is the $\zeta$ function?

Euler was able to show that

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$$Z(4) = \frac{\pi^4}{90}.$$

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This product formula can be used to calculate the probability that a random set of integers are relatively prime.
This probability actually ends up being $\frac{1}{Z(s)}$.

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First, the Gamma function $\Gamma(s)$ is the analytic continuation of the factorial, and was found by Euler. That is, for integer $s > 1$, we have $\Gamma(s) = (s-1)!$.

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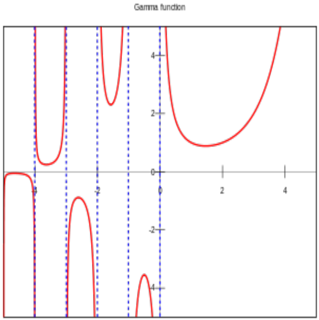
$$\Gamma(s) = \int_0^\infty \frac{dt}{t} t^s e^{-1}, \ \Re(s) > 0.$$



# The Riemann $\zeta$ Function

Consider the integral

$$
\begin{aligned}
L = \int_0^\infty \frac{x^{s-1}}{e^x - 1} dx &= \int_0^\infty \frac{x^{s-1}e^{-x}}{1-e^{-x}} dx \\
&= \sum_{n=0}^{\infty} \int_0^\infty x^{s-1} e^{-(n+1)x} dx \\
&= \sum_{n=0}^{\infty} \frac{\Gamma(s)}{(n+1)^s} \\
&= \Gamma(s)\zeta(s).
\end{aligned}
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$$= \sum_{n=0}^{\infty} \frac{\Gamma(s)}{(n+1)^s}$$

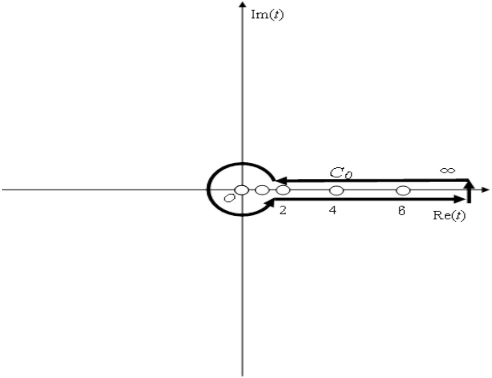
$$= \Gamma(s)\zeta(s).$$

Since we know $L = \Gamma(s)\zeta(s)$, we have

$$\zeta(s) = \frac{L}{\Gamma(s)}.$$

Consider the contour integral

$$K = \int_C \frac{(-z)^{s-1}}{e^z - 1} dz.$$



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Since $\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}$, we get the final form of the $\zeta$ function as

$$\zeta(s) = \frac{\Gamma(1-s)}{2\pi i} \int_C \frac{(-z)^{s-1}}{e^z - 1} dz.$$

# Some Surprising Results

Using this analytic continuation, we can obtain some fascinating results.

$$\zeta(0) = -\frac{1}{2}$$

$$\zeta(-1) = -\frac{1}{12}$$

$$\zeta(-2m) = 0$$

$$\zeta(-3) = \frac{1}{120}$$

$$\zeta(2m) = \frac{(2\pi)^{2m}}{2(2m)!}|B_{2m}|$$

$$\zeta(1-2m) = -\frac{B_{2m}}{2m}$$

# References


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- http://www.math.jhu.edu/~wright/RH2.pdf
- http://mathworld.wolfram.com/RiemannZetaFunction.html
- http://en.wikipedia.org/wiki/Riemann_zeta_function

Thank You!