Weierstrass Factorization Theorem
Name: Hanna Marple, Maria Annone and Mikaila Williams
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1. Introduction to Weierstrass Factorization Theorem:
    a. Entire complex functions can be written as an infinite product of factors involving their zeros
    b. Every infinite sequence has an associated entire function with its zeros at the points of that sequence
    c. Allows us to factor any entire function into a product of zeros and a non-zero entire function

2. Motivation for the Theorem:
    a. For complex polynomials of degree n, you can factor p(z) into n factors associated with their zeros, therefore the sequence of zeros is finite
    b. Can this be extended to all entire functions with an infinite sequence of zeros?
    c. Infinite sequence of zeros imply an infinite product
    d. Weierstrass saw in order for the infinite product of such factors to converge, the individual factors $(z-a_n) \Rightarrow 1$ as the limit of n goes to infinity , where (a_n) is the infinite sequence of zeros
    e. Weierstrass needed functions such that the factors are equal to zero for each specific zero in the sequence, but stay near 1 when not at the specific zero and thus defined Elementary Factors, which possess this property

3. Elementary Factors:
    a. Weierstrass called his new factors, Elementary Factors and defined them as such:
        i. $E_0 = 1 - z$ if n=0
        ii. $E_n = (1 - z)\exp[\frac{z}{1} + \frac{z^2}{2} + \frac{z^3}{3} + \cdots + \frac{z^n}{n}]$otherwise
    b. He defined them this way because he needed factors that converged to 1 as n → infinity and without introducing more zeros to the function

4. The Weierstrass Factorization:
    a. Entire functions with f(z)=0, z DOES NOT equal zero
    b. a_n is an infinite sequence of non-zero complex numbers
        i. $$f(z) = \prod_{n=1}^{\infty} E_{p_n}(z/a_n)$$
        ii. where p_n is a sequence such that for all r>0,

$$\sum_{n=1}^{\infty} (r/|a_n|)^{1+p_n} < \infty.$$

c. Entire functions with f(z)=0, z EQUALS zero

   i. $$f(z) = z^m e^{g(z)} \prod_{n=1}^{\infty} E_{p_n}\left(\frac{z}{a_n}\right)$$

   ii. where m is the multiplicity of z = 0, g(z) is a function, and p_n is a sequence of integers.

5. Example:
   a. $$\sin \pi z = \pi z \prod_{n \neq 0} \left(1 - \frac{z}{n}\right) e^{z/n} = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)$$
   b. $$\cos \pi z = \prod_{q \in \mathbb{Z},\, q \text{ odd}} \left(1 - \frac{2z}{q}\right) e^{2z/q} = \prod_{n=0}^{\infty} \left(1 - \frac{4z^2}{(2n+1)^2}\right)$$

6. Hadamard Factorization:
   a. $$f(z) = z^m e^{g(z)} \prod_{n=1}^{\infty} E_p(z/a_n)$$
   b. generalized form of Weierstrass factorizations
   c. f is an entire function of finite order ρ and g(z) is a polynomial of degree q where q ≤ ρ
   d. where the order of the function is the smallest positive ρ such that for every ε >0, $|f(z)| \leq e^{|z|^{\lambda+\varepsilon}}$

7. Example:
   a. cos (z^(½) is of order ½ and therefore can be factored using Hadamard's method

Sources:

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