Math 360: Fourier Transforms & Differential Equations

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Definitions:

Fourier Transformation:

$$\mathcal{F}(f(x)) = \hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-2\pi i x\xi}dx$$

Inverse Fourier Transformation:

$$\mathcal{F}^{-1}(\hat{f}(\xi)) = f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi)e^{2\pi i x\xi}d\xi$$

Properties:

Linearity

$$(1)\mathcal{F}(f+g) = \mathcal{F}(f) + \mathcal{F}(g) \qquad (2)\mathcal{F}(c\cdot f) = c\cdot\mathcal{F}(f)$$

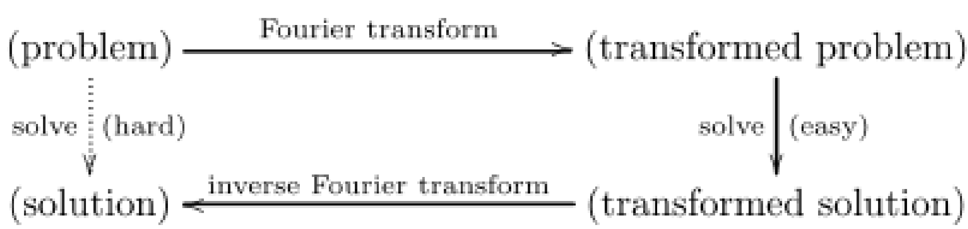
Inverses

$$\mathcal{F}^{-1}(\mathcal{F}(f(x))) = \mathcal{F}(\mathcal{F}^{-1}(f(x))) = f(x)$$

Differentiation Property (proof given in presentation)

$$\mathcal{F}(f'(x)) = (2\pi i\xi)\mathcal{F}(f(x))$$

Using the differentiation property, Fourier Transforms can be used to simplify partial differential equations into ordinary differential equations (or simpler PDE's)



Example: Given the wave equation, $u_{tt} = u_{xx}\ \ \forall\ t > 0,\ x \in (-\infty, \infty)$, we can apply the Fourier Transform to both sides, $\mathcal{F}(u_{xx}(x,t)) = -(2\pi\xi)^2\mathcal{F}(u(x,t))$ and since the transform only switches $x \leftrightarrow \xi$, $\mathcal{F}(u_{tt}(x,t)) = \frac{d^2}{dt^2}\mathcal{F}(u(x,t))$. For simpler notation, we let $v := \mathcal{F}(u(x,t)$, and the problem reduces to

$$v_{tt} = -(2\pi\xi)^2 v$$

which is an ODE with respect to the variable, t. This can be solved using a method of the characteristic equation, assuming v is of the form $e^{\lambda t}$; solving for λ we see $\lambda = \pm(2\pi\xi)$, which solves part of our PDE:

$$\mathcal{F}(u(x,t)) = v = e^{\pm 2\pi\xi t}$$

Using this, we can apply the inverse fourier transform to the equation for our solution.

Notice how the Fourier transform allows us to reduce the PDE to an ODE, which is **MUCH** easier to solve!