Math 360 Complex Variables (Spring 2015) Week 10 Worksheet

Hala A.H. Shehadeh

1 Week 10 Highlights (Midterm Exam on Tues March 17)

1.1 Series Expansions

It is always good to be able to approximate functions locally by polynomials, since polynomials are easy functions to deal with! Complex differentiable functions are so well-behaved in that sense: differentiable functions in a disk can be represented by their ***Taylor series*** and differentiable functions in a punctured disk can be represented by ***Laurent series***.

1.2 Review

1. *Before you start*, review Taylor and Maclaurin series for smooth *real* functions (check the handout on Taylor series from Calcl II). Also check the handout on sequences and series of *real* numbers from Calc II.
2. *What's new for Complex*? Every complex differentiable function in a disk is analytic (can be locally represented by its Taylor series)! That is why, up to this point, we have been using the terms differentiable on a domain, holomorphic, and analytic interchangeably.
3. *Where have we seen series in this class*? We have extensively used the series for e^z. We saw the $\log(z)$ and $\frac{1}{1-z}$ series in worksheet 6. We also used the fact that if $f(z) = 0$ on a segment of a neighborhood of z_0, then it is equal to zero in the whole neighborhood, in order to prove unique determination. This fact is due to Taylor expansion representation of an analytic function.

1.3 Power Series $\sum_{n=1}^{\infty} a_n(z-b)^n$ and Series of Functions $\sum_{n=1}^{\infty} g_n(z)$

We focus on two objects: Power series $\sum_{n=1}^{\infty} a_n(z-b)^n$ and series of functions $\sum_{n=1}^{\infty} g_n(z)$. We need to know for what regions in the complex plane these series converge, what type of convergence, and the properties of the functions they converges to. We should always remember:

1. Any power series $\sum_{n=1}^{\infty} a_n(z-b)^n$ has radius of convergence and a neighborhood of convergence (centered at the point of expansion b). The radius of convergence can be zero, finite, or infinite. You can find the radius of convergence using the ratio or root test from Calculus II. The power series converges point wise in its neighborhood of convergence $|z-b| < R$. Always check separately for convergence on $|z-b| = R$. The power series converges uniformly within $|z-b| \le R_1 < R$.
2. There are different types of convergence: point wise convergence (convergence depends on the point z), absolute convergence (when the series of absolute values $\sum_{n=1}^{\infty} |g_n(z)|$ converges), uniform convergence (convergence is independent of which particular z we choose in the domain). As you go deeper in analysis there are many other types of convergence of sequences of functions in certain spaces.
3. Taylor and Maclaurin series of a function are power series. They have a radius and neighborhood of convergence, can be differentiated and integrated term by term within the neighborhood of convergence. This is due to *uniform convergence*.
4. The Maclaurin series expansions for elementary functions are all the same as in the real case (just replace x with z and replace the intervals of convergence with neighborhoods of convergence).
5. Series expansions break down at points or curves where the represented function is not analytic. These points or curves are *singularities* of the function and studying singularities is important for contour integration, differential equations in the complex plane, and conformal mapping, among others.
6. ***The Weirstrauss M-test*** (this comes in very handy for studying absolute and uniform convergence of series of functions $\sum_{n=1}^{\infty} g_n(z)$): If the terms of the series are bounded by the terms of a convergent series of *positive numbers* on a region R, that is, if $|g_n(z)| < M_n$ for all z in R, and $\sum_{n=1}^{\infty} M_n$ converges, then the series $\sum_{n=1}^{\infty} g_n(z)$ converges uniformly and absolutely in R. (Note that power series do not need this test because we already know everything about them.)
7. We can add and subtract convergent power series term by term. We can also multiply convergent series using the ***Cauchy product***, and compute their reciprocal formally using long division (see page 221 in the book).

1.4 Taylor Series

Let $f(z)$ be differentiable for $|z-b| \leq R$. Then $f(z)$ can be represented by its Taylor series $f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(b)}{n!}(z-b)^n$ in $|z-b| < R$, and the convergence is *uniform* in $|z-b| \leq R_1 < R$. Therefore, f is complex differentiable in a domain D *off* f is *analytic* in D. (The proof is just an application of Cauchy's integral formula and the uniformly convergent expansion $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$.)

1. Maclaurin series is obtained when we expand about the point zero, so $b = 0$ in the above formula.
2. Taylor series representation of an analytic function is unique: There cannot be two Taylor series representations of the same function, also, any convergent power series representation of an analytic function $f(z)$ must be its Taylor series representation.
3. Since Taylor series of a function is unique, Taylor series of some functions can be easily obtained by manipulating Taylor series of some other functions (substitutions for z, integrating or differentiating term by term). For example, see problem 1 in this worksheet.

1.5 Laurent Series

Taylor series cannot be employed in neighborhoods of points at which the function is not analytic. However, another series representation can be found in which both *positive and negative powers* of $(z-b)$ exist. Such a series is valid when a function is analytic in and on a circular annulus $R_1 \leq |z-b| \leq R_2$ (recall that Taylor series is valid on a *disk* $|z-b| \leq R$ on which a function is analytic).
Laurent Series: Let $f(z)$ be analytic in the annulus $R_1 \leq |z-b| \leq R_2$. Then $f(z)$ can be represented by its Laurent series expansion $f(z) = \sum_{n=-\infty}^{\infty} c_n(z-b)^n$ in $R_1 < R_a \leq |z-b| \leq R_b < R_2$, where $c_n = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-b)^{n+1}}dz$, where C any simple closed contour in the region of analyticity enclosing the inner boundary $|z-b| = R_1$. The Laurent series converges uniformly to $f(z)$ for $R_a \leq |z-b| \leq R_b$, and hence can be integrated and differentiated term by term.

1. The coefficient of the term $\frac{1}{(z-b)} = c_{-1}$ is called the ***residue*** of the function $f(z)$ and it plays a very important role in complex analysis. The negative powers of the Laurent series are called the ***principal part*** of $f(z)$.
2. If $f(z)$ is analytic everywhere in the disk $|z-b| < R_1$, then by Cauchy's theorem all the c_n with negative n are zero, and hence the Laurent series becomes the Taylor series of f.
3. Laurent series for $f(z)$ analytic in $R_1 \leq |z-b| \leq R_2$ is unique. This allows us, in practice, to sometimes obtain Laurent series from the Taylor series of a function by appropriate substitutions. For example, replacing z by $\frac{1}{z}$ in the Taylor series of e^z, we get the Laurent series for $e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!z^n}$ which is valid for all $z \neq 0$ (see problems 3 and 4 in this worksheet for more examples).
4. Operations like addition, subtraction, and multiplication of Laurent series behave just like power series.

1.6 Types of Convergence

We've been referring to uniform and absolute convergence, and how uniform convergence allows us to integrate and differentiate a series term by term. The subtlety usually is in interchanging limiting operations (lim and integral, lim and derivative, *etc.*), which is not always valid. However, when a sequence (note that I said sequence, since a series is defined as the limit of the sequence of partial sums) converges uniformly, this interchanging of limits is justified. Review the definitions for point wise, absolute, and uniform convergence, and make sure you know the difference between them.

- It is important to notice that if $\{g_n(z)\}$ is a uniformly convergent sequence of *analytic* functions, we have a much stronger result than we have for a uniformly convergent sequence of only real functions. Namely, *sequences of derivatives of any order* of $g_n(z)$ are uniformly convergent. For example, the *real* sequence $u_n(x) = \frac{\cos(n^2x)}{n}$ where $\infty < x < \infty$ converges uniformly to zero since $|u_n| \leq \frac{1}{n}$ independent of x, however, the sequence of functions $u_n'(x) = 2n\sin(n^2x)$ has no limit whatsoever! (We also note that when we replace x by z, the above sequence $\{u_n(z)\}$ is not uniformly convergent.)

1.7 Recall Analytic Continuation

Reread worksheet 6 and notice how a function can be an analytic continuation of a series beyond its neighborhood of convergence.

1.8 Student Presentation: The Riemann Zeta Function $\xi(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$

(This is analytic for all $x > 1$, where $z = x + iy$.)

2 Reading assignment

Read chapter 5 from the book.

3 Problem Set

Hand the following problems.

1. By manipulating Taylor series that you already know, find the Taylor series expansion about the point $b = 0$ of the following functions in the given regions. Justify all your steps.

   (a) $\frac{z}{1+z^2}$ in $|z| < 1$.

   (b) $\frac{\sin(z)}{z}$ in $0 < |z| < \infty$. Why did we have to delete zero?

   (c) $\log(1+z)$ for $|z| < 1$. (Hint: Integrate term by term the Taylor expansion for $\frac{1}{1+z}$).

2. Find the radius of convergence of the following power series.

   (a) $\sum_{n=1}^{\infty} \frac{z^n}{(n+1)!}$.

   (b) $\sum_{n=1}^{\infty} n^n z^n$.

   (c) $\sum_{n=1}^{\infty} \frac{z^{2n}}{(2n)!}$

3. Expand the function $f(z) = \frac{1}{z(z-1)(z-2)}$ in powers of z valid in:

   (a) $0 < |z| < 1$.

   (b) $1 < |z| < 2$.

   (c) $|z| > 2$.

4. Compute the Laurent expansion of $f(z) = \frac{1}{z(z^2+1)}$ that is valid for:

   (a) $0 < |z| < 1$.

   (b) $1 < |z|$.

5. Show that the two following Laurent expansions are analytic continuations of each other:

$$f(z) = -\frac{1}{z} - 1 - z - z^2 - \ldots 0 < |z| < 1.$$

$$\text{and } g(z) = \frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^4} + \ldots 1 > |z|.$$

6. Let $f(z) = \sqrt{\cos(z)}$, with the branch of square root chosen so that $f(0) = 1$. Consider the power series expansion of $f(z)$ in terms of z.

   (a) Compute the first three nonzero terms of the power series explicitly.

   (b) What is the radius of convergence of this series?

7. (a) For what region in the complex plane does the series $\sum_{n=1}^{\infty} \frac{\sin(nz)}{2^n}$ define an analytic function of z? (Hint: Write $\sin(nz) = \frac{e^{iz} - e^{-iz}}{2i}$.)

   (b) In the region obtained in (a), sum the series to write a formula for the analytic function.

8. Consider the *error function*

$$\text{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt.$$

   This is not an elementary function, however, using series representation of analytic functions, we can analyze a much wider class of functions!

   (a) Using the series representation for the exponential $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$, replacing z by $-t^2$, and integrating term by term (justify why you can integrate this series term by term), write a series expression for the error function $\text{erf}(z)$.

   (b) Deduce that the error function is entire! (Hint: you integrated the exponential function which is entire, or, what is the radius of convergence of your series?)