Math 360 Complex Variables Weeks 13- 14 Worksheet

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1 Weeks 13 and 14 Highlights

1.1 Residue Calculus

Here we deal integrals where the integrand has *isolated singular points*. Each isolated singular point contributes to the integral: a term proportional to the *residue* of the singularity. This is very useful to evaluate definite integrals of various types.

Cauchy's residue theorem: Let $f(z)$ be analytic inside and on a *simple closed contour* C, except for a finite number of isolated singular points $z_1, z_2, \dots, z_N$ located inside C. Then

$$\oint_C f(z)dz = 2\pi i \sum_{n=1}^{N} Res(f(z), z_n).$$

How do we calculate the residues?

1. Recall that the coefficient c_{-1} in the Laurent series expansion of $f(z)$ about the point $z = z_n$ is the residue of $f(z)$. In case the singularity is *essential*, then computing the Laurant expansion is the only general method to evaluate the residue.
2. If the singularity is a *pole* of order m, then $f(z) = \frac{\phi(z)}{(z-z_n)^m}$, where $\phi(z)$ is analytic in the neighborhood of z_n, and $\phi(z_n) \neq 0$. Then the residue $c_{-1} = Res(f(z), z_n)$ is equal to $\frac{1}{(m-1)!}\frac{d^{m-1}\phi(z)}{dz^{m-1}}|_{z=z_n} = \frac{1}{(m-1)!}\frac{d^{m-1}(z-z_n)^m f(z)}{dz^{m-1}}|_{z=z_n}$.
3. If $f(z) = \frac{N(z)}{D(z)}$ where both $N(z)$ and $D(z)$ are analytic in a neighborhood of $z = z_n$, and if $D(z)$ has a zero of order 1 at $z = z_n$, then $c_{-1} = \frac{N(z_n)}{D'(z_n)}$.

1.2 Computing integrals of complex functions

So far, we have 7 methods to evaluate integrals of complex functions:

1. Cauchy's theorem: if the function is analytic in a simply connected domain $\oint_C f(z)dz = 0$.
2. Parametrize the contour C, then $\int_C f(z)dz = \int_a^b f(z(t))z'(t)dt$.
3. Fundamental theorem of calculus: If you already know an antiderivative of the function.
4. Cauchy's integral formula (g is analytic and D is simply connected): $\oint_C \frac{g(\xi)d\xi}{\xi - z_0} = 2\pi i g(z_0)$
5. Generalized Cauchy's integral formula (g is analytic and D is simply connected): $\oint_C \frac{g(\xi)d\xi}{(\xi-z_0)^{n+1}} = \frac{2\pi i}{n!} g^{(n)}(z_0)$.
6. $\oint_C \frac{dz}{z-z_0} = 2\pi i$ if z_0 is in the interior of C, and zero otherwise. (C is a positively oriented loop in simply connected D.)
7. Cauchy's residue theorem (which is an extension of Cauchy's theorem to analytic functions with a finite number of isolated singular points): $\oint_C f(z)dz = 2\pi i \sum_{n=1}^{N} Res(f(z), z_n)$.

1.3 Applications of Residues: Evaluation of certain types of integrals

1. Improper *real* integrals of the form $\int_{-\infty}^{\infty} \frac{p(x)}{q(x)}dx$, $q(z) \neq 0$ on the real line, and the degree of the denominator is at least 2 degrees more than that of the numerator (helps for convergence of the integral over the contour C_R as $R \to \infty$).
2. Improper *real* integrals of the form $\int_{-\infty}^{\infty} f(x)\sin(ax)dx$ or $\int_{-\infty}^{\infty} f(x)\cos(ax)dx$.
   Jordan's Lemma (estimates): Suppose that: $f(z)$ is analytic at all points in the upper half plane that are exterior to a circle $|z| = R_0$; C_R is a semicircle where $R > R_0$; for all points z on C_R there is a positive constant M_R such that $|f(z)| \leq M_R$ and $\lim_{R\to\infty} M_R = 0$, then for every positive constant a, $\lim_{R\to\infty} \int_{C_R} f(z)e^{iaz}dz = 0$. (The proof is based on ***Jordan's inequality*** $\int_0^{\pi} e^{-Rsin\theta}d\theta < \pi/R$ where $R > 0$.)

4. Indented paths: (useful for example, for $\int_{-\infty}^{\infty} \frac{\sin(ax)}{x} dx$)

   Theorem (estimates): Consider an integral on a small circular arc with radius ϵ, center $z = z_0$, and with the arc subtending an angle ϕ. There are two important cases:

   (a) along the contour C_ϵ, $(z - z_0) f(z) \to 0$ uniformly as $\epsilon \to 0$, then $\lim_{\epsilon \to 0} \int_{C_\epsilon} f(z) dz = 0$

   (b) f(z) has a *simple* pole at $z = z_0$ with residue $Res(f(z), z_0) = c_{-1}$, then $\lim_{\epsilon \to 0} \int_{C_\epsilon} f(z) dz = i\phi c_{-1}$, where the integration is carried out in the counterclockwise direction.

5. Indentation around a branch point to avoid a branch point as well as isolated singularities. (Note that a branch point is *not* an isolated singularity). Useful for integrals such as $\int_0^\infty \frac{x^a}{(x^2+1)^2} dx$ where a is a real number $-1 < a < 3$.

6. Integration along a branch cut (useful for integrals such as $\int_0^\infty \frac{\log^2(x)}{x^2+1} dx$, $\int_0^\infty \frac{x^a}{x+1} dx$ where $0 < a < 1$, $\int_{-1}^{1} \frac{\sqrt{1-x^2}}{1+x^2} dx$, *etc.*.)

   Theorem (estimates): This may be useful for providing estimates for cases where Jordan's lemma is not applicable: If on a circular arc of radius R and center $z = 0$ we have $z f(z) \to 0$ uniformly as $R \to \infty$, then $\lim_{R \to \infty} \int_{C_R} f(z) dz = 0$.

7. Integrals of the form $\int_0^{2\pi} g(\sin\theta, \cos\theta) d\theta$.

1.4 Residue at infinity

2 Reading assignment

Read chapter 6 (up to page 237) and chapter 7 (up to page 286) from the book.

3 Problem Set

Problems from the book:

1. Number 5 page 238.
2. Number 6 page 238.
3. Number 5 page 265.
4. Number 9 page 265.
5. Number 4 page 273.
6. Number 12 page 273.
7. Number 2 page 282.
8. Number 2 page 287.
9. Number 5 page 287.

Outside problems:

10. Use contour integration and residues to compute the integral $I = \int_{-1}^{1} \frac{\sqrt{1-x^2}}{1+x^2} dx$.

11. (a) Show that on the lines $\Re(z) = N + \frac{1}{2}$, where N is a positive integer, the functions $\frac{\cos(\pi z)}{\sin(\pi z)}$ and $\frac{1}{\sin(\pi z)}$ are uniformly bounded.

    (b) Evaluate $\frac{1}{2\pi i} \int_{C_N} \frac{\cos(\pi z)}{\sin(\pi z)(z^2+1)} dz$ and $\frac{1}{2\pi i} \int_{C_N} \frac{1}{\sin(\pi z)(z^2+1)} dz$, where C_N is the square with vertices $\pm(N + \frac{1}{2}) \pm i(N + \frac{1}{2})$. The contour is traversed in the counterclockwise direction.

    (c) Sum the series $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ and $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+1}$.