Math 360 Complex Variables (Spring 2015) Week 15 Worksheet

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1 Week 15 Highlights

1.1 Isolated singular points

A singular point z_0 of $f(z)$ is isolated if $f(z)$ is analytic in a neighborhood of z_0 but not at z_0. In the neighborhood of a singular point, we proved that $f(z)$ has a Laurent series representation

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n = \ldots + \frac{c_{-2}}{(z - z_0)^2} + \frac{c_{-1}}{(z - z_0)} + \sum_{n=0}^{\infty} c_n (z - z_0)^n.$$

The principal part of the Laurent series (part with negative n's) determines the types of the isolated singularity.

1. **Removable singularity**
   The Laurent series near a removable singularity has no principal part! Example: $z = 0$ is a removable singularity for $f(z) = \frac{\sin z}{z} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \ldots$. Note that the function is not defined at $z = 0$, but its Laurent series is, so we can redefine the function to be 1 at $z = 0$, and hence we are able to 'remove' the singularity.

   - The residue $Res(f(z), z_0) = c_{-1}$ at a removable singular point is always zero.
   - Behavior near removable singularity ***theorem:*** If z_0 is a removable singularity of $f(z)$ then $f(z)$ is *bounded* and analytic in some deleted neighborhood of z_0.
   - Behavior near removable singularity ***Riemann's theorem:*** Suppose that $f(z)$ is *bounded* and analytic in some deleted neighborhood $0 < |z - z_0| < \epsilon$ of z_0. If f is not analytic at z_0, then it has a removable singularity there.

2. **Pole**
   The principal part of the Laurent series only has finitely many nonzero c_n's, and the order of the pole is $m = |n|$ where n is the negative integer largest in magnitude. A pole of order one is called a *simple* pole. Example: The function $\frac{1}{z^2(1-z)} = \frac{1}{z^2}(1 + z + z^2 + z^3 + \ldots) = \frac{1}{z^2} + \frac{1}{z} + 1 + z + z^2 + \ldots$ has a pole of order 2 at $z = 0$. Another way to identify poles: Suppose $p(z)$ and $q(z)$ are analytic at the point z_0, $p(z_0) \neq 0$, and $q(z)$ has a zero of order m at z_0, then $\frac{p(z)}{q(z)}$ has a pole of order m at z_0 (so the above function $\frac{1}{z^2(1-z)}$ has a pole of order 2 at $z = 0$ and a simple pole at $z = 1$).

   - Behavior near pole, ***theorem:*** If z_0 is a pole of a function $f(z)$, then $\lim_{z \to z_0} f(z) = \infty$. (Note that this is different than the case where z_0 is a removable singularity, since $f(z)$ is bounded there, and this limit is not infinite.)

3. **Essential singularity**
   An essential singularity has a "full" Laurent series, meaning the principal part has infinitely many nonzero c_n's. Example $e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!}\frac{1}{z^2} + \ldots$ has an essential singularity at $z = 0$.

   - Behavior near essential singularity, ***Picard's theorem***: In each neighborhood of an essential singularity, a function assumes *every* finite value, with one possible exception, an *infinite* number of times.
   - Behavior near essential singularity, ***Casorati-Weierstrass theorem***: If $f(z)$ has an essential singularity at $z = z_0$, then for any complex number w, $f(z)$ becomes arbitrary close to w in a neighborhood of z_0. That is, given w, and any $\epsilon > 0$, $\delta > 0$, there is a z such that $|f(z) - w| < \epsilon$ whenever $0 < |z - z_0| < \delta$.

1.2 Non-isolated singular points

For example, a branch point is a non-isolated singular point. Another example is a *cluster* singular point, which happens when an infinite sequence of isolated singular points cluster around a point (for example, $z = 0$ is a cluster singular point for $f(z) = \tan(1/z)$). There is no valid Laurent series representation in the neighborhood of a cluster singular point.

1.3 Meromorphic function

$f(z)$ is meromorphic in a domain D is it is analytic throughout D except for poles.

1.4 Winding number

$\frac{1}{2\pi}\Delta_C arg(f(z))$ represents the number of times the image γ of a simple closed contour C in the z-plane winds around the origin in the w-plane. It is easy to see that the winding number is an integer (it could be zero if γ winds zero times around the origin). Here, $f(z)$ is meromorphic in the domain interior to positively oriented C, and analytic and nonzero *on* C (so γ does not pass through the origin).

1.5 Argument principle

For a simple closed contour C and a meromorphic function $f(z)$ inside C, the winding number is simply the difference $Z - P$ between the number of zeros of $f(z)$ and its poles inside C, counting multiplicities for both the zeros and the poles. (The proof uses the fact that

$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}dz = \text{winding number},$$

(this happens when we parametrize $f(z) = \rho(t)e^{i\phi(t)}$). On the other hand, using Cauchy's residue theorem, we can show that the above integral is in turn equal to $Z - P$.) Note that $f(z)$ has to be meromorphic in C, analytic and nonzero *on* C.

1.6 Rouche's theorem

This is a consequence of the argument principle, and is useful for locating regions of the complex plane where a given analytic function has zeros. It mainly says that inside a simple closed contour, an analytic function has as many zeros as its 'dominant' part. So we can only count the zeros of the dominant part of the function, instead of dealing with the whole function, which is usually more complicated.
Rouche's theorem: Let $f(z)$ and $g(z)$ be analytic inside and on a simple closed contour C, and let $|f(z)| > |g(z)|$ at each point *on* C, then $f(z)$ and $f(z) + g(z)$ have the same number of zeros, counting multiplicities, inside C. (The proof follows from applying the argument principle to the function $1 + \frac{g(z)}{f(z)}$, which has winding number zero because of the hypothesis.)

2 Reading assignment

Read the rest of chapters 6 and 7 from the book.

3 Problem Set

Hand the following problems.

3.1 Problems from the book

1. Page 293 number 2.
2. Page 294 number 7.
3. Page 294 number 8.

3.2 Problems from outside the book

4. Discuss the location and the type of singularities (if it is a pole, give the order) of the following functions. State whether the singularity is isolated or not:
   (a) $f(z) = \frac{e^{z^2}-1}{z^2}$.
   (b) $g(z) = \frac{(z-1)^2}{z(z+1)^3}$.
   (c) $h(z) = \frac{z+1}{z\sin(z)}$.
5. Show that the function $h(z) = e^z - 4z - 1$ has exactly one root inside the unit circle $C : |z| = 1$, and that the function $g(z) = e^z - 4z^2 - 1$ has exactly 2 roots inside the unit circle.
6. Let $f(z) = 2z^4 + 3z^2 - 2z + 1 + \frac{9}{z}$. Show that $f(z)$ has 5 zeros in the annulus $1 < |z| < 2$.
7. Prove that if $A > 1$ and n is a positive integer, the equation $z^n e^{A-z} = 1$ has exactly n roots inside the unit circle (Hint: rewrite the equation as $z^n - e^z e^{-A} = 0$ and apply Rouche's theorem). Show also that for $n = 1$, this root is real and positive (Hint: for $h(z) = ze^{A-z} - 1$, compute $h(0)$ and $h(1)$, then use the continuity of h).