Math 360 Complex Variables (Spring 2015) Week 16 Worksheet

Hala A.H. Shehadeh

1 Week 16 Highlights

1.1 Conformal Mapping

Theorem (Conformal Mapping): A map $w = f(z)$ is conformal (angle preserving) at z_0 *iff* f is analytic at z_0 and $f'(z_0) \neq 0$. In fact, if f is conformal at z_0, then it is conformal at each point in some neighborhood of z_0.
A map f is conformal in a domain D *iff* it is conformal at each point in D.
Conformal mapping provides a classical method for solving problems in continuum mechanics, electrostatics, and other fields involving *two-dimensional* Laplace ($\phi_{xx} + \phi_{yy} = 0$) and Poisson ($\phi_{xx} + \phi_{yy} = g$) equations.
A Boundary Value Problem looks like

$$\begin{cases} \phi_{xx} + \phi_{yy} = 0 \text{ in } D \\ \phi = \phi_0 \text{ (Dirichlet problem) on } \partial D \text{ or } \dfrac{\partial \phi}{\partial n} = 0 \text{ (Neumann problem) on } \partial D. \end{cases}$$

We are usually interested in finding solutions of these problems in a certain domain (than has an ugly shape) with certain boundary conditions (prescribed values of the unknown function ϕ and/or its derivatives on some portions of the boundary of the ugly domain).
These problems are greatly simplified if we can transform the ugly domain into a domain which is easy to deal with, like: the upper half plane, the unit disk, the horizontal strip $0 < y < \pi$, or the first quadrant. Since we are interested in *boundary value* problems, we also need to transform the *boundary conditions* as well! The following two theorems allow us to do so.

1.2 Transforming Boundary Value Problems

1. ***Transforming harmonic functions*** The heart of this theorem: *the composition of a harmonic function with an analytic function is harmonic*! Statement: If $f(z) = w = u(x, y) + iv(x, y)$ maps a domain D_z in the z-plane onto a domain D_w in the w-plane, and the function $h(u, v)$ is harmonic in D_w (satisfies Laplace equation), then the function $H(x, y) = h \circ f(x, y) = h(u(x, y), v(x, y))$ is harmonic in D_z.

   Note that f only needs to be analytic. So using this theorem, we can map the difficult domain to the easy domain using f, then find a harmonic function $h(u, v)$ in the easy domain (note that the real and imaginary parts of analytic functions provide an unlimited supply of harmonic functions), then finally find the real and imaginary parts $u(x, y)$ and $v(x, y)$ of f, and plug them back into h. This given us the harmonic function in the difficult domain.

   The following theorem guarantees that the Dirichlet and Neumann boundary conditions remain intact when we conformally map the difficult domain to the easy one:

2. ***Transforming Dirichlet and Neumann boundary conditions*** Let $f(z) = w = u(x, y) + iv(x, y)$ be conformal at each point of a smooth arc C and that γ is the image of C under f. Let $h(u, v)$ be a function that satisfies either the Dirichlet condition $h = h_0$ (a real constant) or the Neumann condition $\frac{\partial h}{\partial n} = \nabla h.\mathbf{n} = 0$ (directional derivative of h normal to γ.) Then the function $H(x, y) = h \circ f(x, y) = h(u(x, y), v(x, y))$ satisfies exactly the same conditions ($H = h_0$ or $\frac{\partial H}{\partial N} = 0$) on C.

1.3 The Riemann Mapping Theorem

A corollary of the Riemann mapping theorem (below) guarantees that any two *simply connected domains* which are not the whole complex plane can be mapped conformally onto each other! We need this in order to be able to map the ugly domains D above conformally onto the good domains where we can find harmonic functions with the given boundary conditions.
Riemann Mapping Theorem: Let z_0 be a point in a simply connected domain D which is not all of the complex plane. Then there is a unique analytic function $w = f(z)$ mapping D one-to-one onto the open unit disk $\{|w| < 1\}$ such that $f(z_0) = 0$ (so z_0 is mapped onto the origin) and $f'(z_0)$ is real and positive.

1.4 Table of Transformations

We obviously need a glossary of transformations that come in handy for transforming the two dimensional domains that appear in our boundary value problems. Of course, these maps are guaranteed to exist by the Riemann mapping theorem (however, the proof of the Riemann mapping theorem is nonconstructive, so the proof itself does not tell us how to find these mappings). Check Appendix 2 in the back of the book.

1.5 The Schwarz-Christoffel Transformation

This is an extremely useful transformation that maps the x-axis and the upper half plane in the z-plane onto a given simple closed polygon and its interior in the w-plane. As we said before, this is useful because solving Laplace equation (to get harmonic functions) in the upper half plane with prescribed boundary conditions is a significantly easier problem than solving it in a polygon. (For details, see chapter 11 in the book.)

2 Reading assignment

Read chapters 9 and 8 (in this order, 9 then 8) from the book (for applications, chapter 10 is a great read). Check out Appendix 2 for the table of transformations.

3 Problem Set

One of these problems will be on the final exam.

1. Find the electrostatic potential V in the space enclosed by the half circle $x^2 + y^2 = 1, y \geq 0$ and the line $y = 0$ when $V = 0$ on the circular boundary, and $V = 1$ on the line segment $[-1, 1]$.

2. (a) Find a map $w = f(z)$ that maps the right half plane $\Re(z) > 0$ into the interior of the unit circle $|w| < 1$, so that the points $z = 0, i, \infty$ are mapped to the points $w = -1, i, 1$ respectively.

   (b) Use the result to find a harmonic function $\phi(r, \theta)$ as a function of polar coordinates (r, θ), defined within the unit disk, and taking the values 1 on the upper hand and -1 on the lower half of the unit circle $|w| = 1$ (these are the boundary values). (Hint: It is probably easier to map the horizontal strip $0 \leq y \leq \pi$ to the right half plane first.)

3. The famous ***Joukowski airfoil*** (think of the effect of the transformation $w = z + \frac{1}{z}$ on a circle): Number 15 page 391 from the book.