Math 360 Complex Variables (Spring 2015) Week 2 Worksheet

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1 Week 2 Highlights

1.1 More on complex numbers and the complex plane

1. *Roots of complex numbers* ($z \to z^{1/n}$ is a muti-valued function from $\mathbb{C} \to \mathbb{C}$): use polar representation, $z^{1/n} = r^{1/n}e^{i\frac{\theta+2k\pi}{n}}, k = 0, 1, \ldots, n-1$. Special case: n-th roots of unity $\sqrt[n]{1}$ in $\mathbb{C}$.
2. *Important regions in the complex plane*: open disk of center z_0 and radius ϵ, $D_\epsilon(z_0) = \{z : |z - z_0| < \epsilon\}$; closed disk $\bar{D}_\epsilon(z_0)$; open set U; neighborhood of z_0 is any open set containing z_0; boundary of a region D is $\bar{D} \cap D^C$.

1.2 Complex functions

We can now introduce complex functions of complex numbers! Continuity and differentiability are measures of the *niceness* of our functions. Differentiability later will be equivalent to analyticity. Recall that in our Calculus classes, we studied functions of real variables, their continuity, differentiability and integrability. We will do the same here, and experience the strength of complex differentiability, and later see that complex differentiability is in some sense not at all different than complex integrability!

1. *Complex valued functions*: $f : \mathbb{C} \to \mathbb{C}$ ($x + iy \longmapsto u(x, y) + iv(x, y)$, and visualizing in the z and ω planes).
2. *Continuity*: $f : \mathbb{C} \to \mathbb{C}$ is continuous at z_0 iff given $\epsilon > 0$, there exists a $\delta > 0$ such that $|f(z) - f(z_0)| < \epsilon$ for all $|z - z_0| < \delta$. (Visualize this graphically in the z and ω planes.)
3. Examples of functions we've seen so far on $\mathbb{C}$:
   (a) $f(z) = \bar{z} : \mathbb{C} \to \mathbb{C}$ (not differentiable anywhere on $\mathbb{C}$).
   (b) $f(z) = |z|^2 : \mathbb{C} \to \mathbb{C}$ (differentiable only at zero, which is a rarity in calculus for real valued functions- can you think of a real valued function that is differentiable only at one point?).
   (c) $f(z) = z^n : \mathbb{C} \to \mathbb{C}$ (*entire*, which means differentiable over the whole complex plane.)
   (d) $f(z) = \frac{1}{z^n} : \mathbb{C} \to \mathbb{C}$ (differentiable everywhere except for a singularity at zero, however we'll see later it's not a *bad* singularity.)
   (e) $f(z) = \mathrm{Arg}z : \mathbb{C} \to (-\pi, \pi]$ (discontinuous along the negative real axis.)
   (f) $f(z) = \arg z : \mathbb{C} \to \mathbb{R}$: multivalued function, assigns to each z infinitely many *arguments* $\theta + 2k\pi$.
   (g) $f(z) = z^{1/n} : \mathbb{C} \to \mathbb{C}$: multivalued function, assigns n distinct roots to each z, corresponding to each of $k = 0, 1, 2, \ldots, n-1$ in $r^{1/n}e^{i\frac{\theta+2k\pi}{n}}$.

1.3 Complex Differentiability

1. *Differentiability*: $f : \mathbb{C} \to \mathbb{C}$ is differentiable at z_0 iff

$$f'(z_0) = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

exists. Note that since we are in the complex plane, h can approach zero from any direction (infinitely many ways to approach zero), and the limit should exist and have the same value *independent* of which direction $h \to 0$. This makes differentiability in the complex plane a *nontrivial* property, and a complex differentiable function is a very well-behaved function with many nice properties.

   (a) A function which is differentiable on an open subset U of the complex plane is called *holomorphic*, *regular* or later *analytic* on U.

   (b) A function which is differentiable over the entire complex plane is called *entire* in $\mathbb{C}$.

2. *Cauchy Riemann equations*: There is an interlocking between the real and imaginary parts of a complex differentiable function: f is differentiable at $z_0 = x_0 + iy_0$ iff $u_x = v_y$ and $u_y = -v_x$ at (x_0, y_0), also $f'(z) = u_x + iv_x = v_y - iu_y$ evaluated at z_0.

   The Cauchy Riemann equations are in fact equivalent to $\frac{\partial f}{\partial \bar{z}} = 0$! So you can substitute $x = \frac{z+\bar{z}}{2}$ and $y = -i\frac{z-\bar{z}}{2}$, and if the $\bar{z}$ cancels out, then your function is holomorphic, without having to go through the trouble of computing all the derivatives involved in the Cauchy Riemann equations.

3. *Harmonic Functions*: A real valued function $h : \mathbb{R}^2 \to \mathbb{R}$ is harmonic on a domain D of the xy-plane if, throughout that domain, it has continuous partial derivatives of first and second order, and it satisfies *Laplace's equation*

$$h_{xx}(x, y) + h_{yy}(x, y) = 0.$$

   (a) If $f(x + iy) = u(x, y) + iv(x, y)$ is analytic on D, then $u(x, y)$ and $v(x, y)$ are both harmonic on D, and v is called the *harmonic conjugate* of u.

   (b) If v is the harmonic conjugate of u it doesn't necessarily imply that u is the harmonic conjugate of v! In fact, v is the harmonic conjugate of u *iff* $-u$ is the harmonic conjugate of v.

   (c) *(Weak) Maximum principle for harmonic functions:* A non-constant harmonic function on a domain cannot have a local maximum or minimum within the domain, and must achieve its minimum and maximum values on the boundary of the domain. Hence a harmonic function is bounded inside a domain by its values on the boundary. Use *Matlab* to plot the real and imaginary parts of analytic functions and see that they actually satisfy this property!

   (Later we will learn about the *maximum modulus principle for analytic functions*!)

   (d) Mean value property for harmonic functions.

   (e) Harmonic functions arise in many applications. Later in this class, they will be useful in potential theory, e.g., in the Newtonian theory of gravity, electrostatics, heat flow, and potential flow in fluid mechanics.

2 Reading assignment

Read Chapter 2 from the book (except the sections on uniquely determined analytic functions and the reflection principle).

3 Problem Set

Hand the following problems.

1. Show that if z lies on the circle |z|=2, then
$$|z^4 - 4z^2 + 3| \geq 3.$$

2. Find the principal argument $Arg z$ for
$$z = \frac{i}{-2-2i}.$$

3. Using the polar form for $z = re^{i\theta}$, show that
$$(-1+i)^7 = -8(1+i).$$

4. Find all the roots of $8^{1/6}$ in $\mathbb{C}$, and plot them in the z-plane.

5. Find the four roots of the equation $z^4 + 4 = 0$, and use them to factor $z^4 + 4$ into *quadratic* factors with *real* coefficients.

6. Sketch the following sets in the z-plane, then determine whether the set is open, closed, or neither open or closed, also determine whether the set is bounded or unbounded.

   (a) $|z - 2 + i| \leq 1$.

   (b) $\Im(z) > 1$.

   (c) $0 \leq \arg z \leq \pi/4 (z \neq 0)$.

   (d) $|z - 4| \geq |z|$.

7. What is the domain of definition of the following functions on $\mathbb{C}$?

   (a) $f(z) = \frac{1}{z^2+1}$

   (b) $f(z) = Arg\left(\frac{1}{z}\right)$.

   (c) $f(z) = \frac{1}{z^4}$

8. Write $f(x+iy) = x^2 - y^2 - 2y + i(2x - 2xy)$ as a simple expression in terms of z and $\bar{z}$. Is this function analytic (holomorphic)? Why or why not?

9. Show that the following functions are *entire*, and find $f'(z)$ and $f''(z)$.

   (a) $f(x+iy) = e^{-x}e^{-iy}$.

   (b) $f(z) = z^3$.

10. Show that the function $u(x,y) = \frac{y}{x^2+y^2}$ is harmonic everywhere in $\mathbb{R}^2$ except at zero, and find its harmonic conjugate $v(x,y)$. Use Matlab to demonstrate the maximum principle for both u and v. Finally, express the analytic (holomorphic) function $f(x+iy) = u(x,y) + iv(x,y)$ that you obtained in terms of z, and find $f'(z)$.