Math 360 Complex Variables (Spring 2015) Week 3 Worksheet

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1 Week 3 Highlights

1.1 More on harmonic functions

1. The level curves of a harmonic function $u(x, y)$ and its harmonic conjugate $v(x, y)$ are orthogonal.
2. Liouville's theorem (later, compare with Liouville's theorem for complex analytic functions): A *bounded* harmonic function on *all* of $\mathbb{R}^2$ (or $\mathbb{R}^n$) is constant.
3. Locally, any harmonic function is the real part of a complex analytic function.
4. Application of harmonic functions and their harmonic conjugates (potential field and fluid flow).

1.2 Elementary Functions

Now we study the complex analogues of the elementary functions we've encountered in Calculus. These can have drastically different properties than their real counterparts.

1. Polynomial functions $p(z)$ (entire). Eventually, we will be able to approximate every complex analytic function by a polynomial, at least locally, if we allow the polynomial to have arbitrary high degree (Taylor series representation.)

   Fundamental theorem of algebra: every non-zero, single-variable, degree n polynomial with complex coefficients has, counted with multiplicity, exactly n roots in $\mathbb{C}$. (It has 3 major proofs, we'll see the complex analysis proof later.)
2. Rational functions $f(z) = \frac{p(z)}{q(z)}, q(z) \neq 0$ (analytic away from the zeros of $q(z)$, and the set $\mathbb{C} \setminus \{z : q(z) = 0\}$ is open). The zeros of $q(z)$ are called *poles* of f. Using the limit definition of the derivative we have, just as in $\mathbb{R}$, $\left(\frac{p(z)}{q(z)}\right)' = \frac{p(z)'q(z) - p(z)q(z)'}{q(z)^2}$.
3. Exponential function e^z. The most important function in mathematics! We'll spend some time on it next week. The complex exponential function is *periodic* and that is different than one-to-one real e^x. This periodicity causes the complex logarithm $\log z$ to be multivalued.
4. Trigonometric functions $\cos z$, $\sin z$, *etc.*. Contrary to their real restrictions, the complex cosine and sine functions are *not* bounded (their real and imaginary parts have cosh and sinh in them, which are unbounded).
5. Complex hyperbolic functions $\cosh z$, $\sinh z$, *etc.*: Contrary to their real counterparts, these are not independent from the complex trigonometric functions.
6. Algebraic functions $z^{p/q}$, *etc*: We studied roots of complex numbers and the multivalued-ness of these functions.
7. The complex logarithm $\log z$: This is defined for all complex numbers except zero, which is a branch point for $\log z$ (including the negative real axis, which is not the case for real $\log x$), but at a price! It is not single valued, and the need to specify an appropriate branch and branch cut arises.
8. Inverse trigonometric functions $\sin^{-1} z$, $\cos^{-1} z$, *etc.*.

1.3 Other interesting functions

The above list is far from exhaustive and there are many other interesting functions that we encounter in $\mathbb{R}$ and can be generalized to $\mathbb{C}$, and they may retain some of their properties on $\mathbb{R}$ or have completely different ones. For example,

1. Gamma function.
2. Airy functions.
3. Bessel functions.
4. Legendre functions.
5. Elliptic functions.
6. Modular functions.
7. Riemann zeta function (we will see this function later in this class.)

1.4 Student presentation on History of Complex Analysis.

2 Reading assignment

Read Chapter 3 from the book (same reading assignment for Week 4 as well).

3 Problem Set

Hand the following problems.

1. Let $f(z) = \frac{1}{1+z^6}$.
   (a) Find, and plot in the z-plane, the poles of f.
   (b) Write $f(z) = g(z)h(z)$, where g is analytic in the lower half plane, and h is analytic in the upper half plane, so that $g(z) = \overline{h(\overline{z})}$.
2. Consider the function $f(z) = z^2$.
   (a) Find its real and imaginary parts $u(x, y)$ and $v(x, y)$.
   (b) Use Matlab to plot the level curves of u and v (contour plot), and observe the orthogonality of these curves.
   (c) The curves $u(x, y) = 0$ and $v(x, y) = 0$ intersect at the origin, however, they are not orthogonal to each other. Why is this still O.K.?
3. Same problem as above (only parts (a) and (b)) for $f(z) = \frac{z-1}{z+1}$. (To find the real and imaginary parts, it is easy to multiply up and down by the conjugate $\bar{z}+1$ of the denominator, so the denominator easily becomes $|z + 1|^2 = (x + 1)^2 + y^2$.)
4. Use polar representation $z = re^{i\theta}$ to explain how the function $f(z) = z^2$ maps the first quadrant in the z-plane onto the upper half plane in the w-plane, in a one-to-one manner. Show also that the same transformation maps the upper half plane onto the entire w-plane, however, the transformation here is not one-to-one.