Math 360 Complex Variables (Spring 2015) Weeks 4 and 5 Worksheet

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1 Weeks 4 and 5 Highlights

1.1 Composition

of analytic functions is analytic, and $(f \circ g)'(z) = f'(g(z))g'(z)$. (Here, g is analytic on an open subset U of $\mathbb{C}$, and f is analytic on an open subset containing $g(U)$.) Note that since U is open and g is analytic, then $g(U)$ is also open. This is ***the open mapping property*** of analytic functions.

1.2 Elementary functions: More details

In the following, $z = x + iy = |z|e^{i\arg(z)} = |z|e^{iArg(z)+2\pi ik}, k = 0, \pm 1, \pm 2, \ldots$.

1. The **complex exponential** e^z (single-valued, periodic, entire, never zero, assumes all complex values except zero).
   (a) $e^z = e^x e^{iy} = e^x \cos(y) + ie^x \sin(y)$.
   (b) e^z is entire and $(e^z)' = e^z$ (Cauchy Riemann equations and the derivative in terms of u and v).
   (c) $e^{z_1}e^{z_2} = e^{z_1+z_2}$.
   (d) $e^z \neq 0$ for any complex number z (since $e^z e^{-z} = e^0 = 1$).
   (e) e^z is a periodic function on $\mathbb{C}$ with period $2\pi i$ (so $e^{z+2k\pi i} = e^z, k = 0, \pm 1, \pm 2, \ldots$). (This is different than the restriction e^x of the exponential to $\mathbb{R}$, which is one-to-one, so it's invertible with inverse $\ln(x)$.) This periodicity will imply that the complex logarithm will not be single-valued! We can divide up the complex plane into horizontal strips of height 2π each, so that the complex exponential is one-to-one function on each strip. Note that each strip gets mapped onto the whole complex plane minus zero under e^z.
   (f) If w is a *nonzero* complex number, then $w = e^z$ for some z (so e^z assumes all values in $\mathbb{C} \setminus \{0\}$).
   (g) All the above properties, including the definition (a), can also be proved if we define the exponential using the series $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$.
2. **Complex trigonometric functions**
   (a) $\cos(z) = \frac{e^{iz}+e^{-iz}}{2}$ and $\sin(z) = \frac{e^{iz}-e^{-iz}}{2i}$.
   (b) $\cos(z)$ and $\sin(z)$ are entire, and, $\cos'(z) = -\sin(z)$ and $\sin'(z) = \cos(z)$.
   (c) Many of the properties of the real $\sin(x)$ and $\cos(x)$ functions carry over to their complex extensions, for example $\cos^2(z) + \sin^2(z) = 1$, however, the complex sine and cosine functions are not bounded (unlike real sine and cosine which are bounded by 1)! The unboundedness is due to having the functions sinh and cosh in their real and imaginary parts:
   $$\sin(x + iy) = \sin(x)\cosh(y) + i\cos(x)\sinh(y),$$
   $$\cos(x + iy) = \cos(x)\cosh(y) - i\sin(x)\sinh(y).$$
   In fact, if an entire function is bounded, it must be a constant! This is ***Liouville's theorem*** which we will prove later. So complex sine and cosine couldn't be entire and bounded.
   (d) Define other trigonometric functions like $\tan(z)$ (note that $\tan z$ has poles at odd multiples of $\pi/2$), *etc.* in the usual way, and make yourself familiar with their properties.

3. **Complex hyperbolic functions**

   (a) $\cosh(z) = \frac{e^z+e^{-z}}{2}$ and $\sinh(z) = \frac{e^z-e^{-z}}{2}$ (entire, find their derivatives).

   (b) These, unlike their real restrictions, are not independent from the complex trigonometric functions: $\sinh(iz) = i\sin(z)$ and $\cosh(iz) = \cos(z)$.

   (c) There are many other properties (periodic, with period $2\pi i$), which you should make yourself familiar with.

4. **The complex logarithm** $\log(z)$ (multi-valued because it's the inverse of periodic e^z, defined for *all* complex numbers except zero, analytic only after we specify a branch, introduce *branch cuts*).

   (a) $\log(z) = w \Longleftrightarrow z = e^w$ (recall that $e^{w+2k\pi i} = e^w$).

   (b) $\log(z) = \ln(|z|) + i\arg(z)$ (but $\arg(z)$ is multivalued, so is $\log(z)$).

   (c) Contrary to real valued $\ln(x)$ on $\mathbb{R}$, we can take $\log(-1) = \ln|-1| + i\arg(-1) = i(2k+1)\pi$, $k = 0, \pm1, \pm2, \ldots$.

   (d) Branches, branch cuts, and branch points: Making $\log(z)$ single-valued comes at a price- we lose continuity over the whole complex plane. To do this, we specify a *branch* $\mathrm{Log}_\tau(z)$ of $\log(z)$, by restricting the argument $\tau \le \mathrm{Arg}_\tau < \tau + 2\pi$. Log is discontinuous across the branch cut $\theta = \tau$. (So for $\tau = -\pi$, the principal branch $\mathrm{Log}(z)$ is discontinuous along the negative real axis.). A point common to all branch cuts is called a *branch point* (In fact, a point is a ***branch point*** if the multivalued function is discontinuous upon traveling a short circuit around this point.). So the origin is a branch point for $\log(z)$ (the point at infinity $z = \infty$ is also a branch point).

   (e) $\mathrm{Log}_\tau(z)$ is analytic over $\mathbb{C} \setminus \{\text{the branch cut } \theta = \tau \text{ and the origin}\}$ and its derivative is $\mathrm{Log}_\tau(z)' = \frac{1}{z}$.

   (f) There is usually no benefit in choosing one branch of the log over another, but one should be consistent the moment a choice is made. Some choices may be more convenient than other for specific applications, for example, to compute the derivative of $\log(z)$ at the point -1, the principal branch would not be a valid choice, since it is not analytic there, but any other valid branch would be fine, and all valid choices would give the same result.

   (g) Algebraic identities like $\log(z_1z_2) = \log(z_1) + \log(z_2)$ and $\log(z_1/z_2) = \log(z_1) - \log(z_2)$ still hold up to integer multiples of $2\pi i$, since $\log(z)$ is multivalued. (Always check whether an identity is still valid.)

5. **Complex powers** z^α (α is a real or a complex number) (inherits multi-valued property from the complex logarithm, so need to specify branch for analyticity).

   (a) $z^\alpha = e^{\alpha\log(z)} = e^{\alpha(\ln|z|+i\arg(z))}$ (composition of two functions which we already defined). (Note that if α is an integer, this function is single valued; if α is a rational number p/q, then this function has finitely many values; and if α is an irrational number or a complex number (with nonzero imaginary part), then this function has infinitely many values!)

   (b) Every branch of $\log(z)$ gives rise to a branch of z^α. Since the exponential is entire, z^α is is analytic in the domain where the specified branch of $\log(z)$ is analytic, and its derivative $(z^\alpha)' = \alpha z^{\alpha-1}$.

   (c) Algebraic identities: some still hold as in the real case, like $z^\alpha z^\beta = z^{\alpha+\beta}$, others don't, for example $z_1^\alpha z_2^\alpha \neq (z_1z_2)^\alpha$ for nonzero complex z's and α!

   (d) z^z is an analytic function whenever a branch of the logarithm is chosen and analytic, and $(z^z)' = z^z(\mathrm{Log}_\tau(z) + 1)$.

6. **Inverse trigonometric functions** (inherits multi-valued property from the complex logarithm, so need to specify branch for analyticity).

   (a) Define in terms of logarithm: $\sin^{-1}(z) = w \iff z = \sin(w) = \frac{e^{iw}-e^{-iw}}{2i}$, then solving a quadratic equation for e^{iw}, we get

   $$\sin^{-1}(z) = -i\log(iz + (1-z^2)^{1/2}).$$

   This is a multivalued function with infinitely many values at each point z. Note that the function $iz + (1-z^2)^{1/2}$ has two branches, and the log function has infinitely many branches, hence $\sin^{-1}(z)$ is said to be *doubly infinite.*

   (b) Find similar expressions for $\cos^{-1}(z)$, $\tan^{-1}(z)$, $\sinh^{-1}(z)$, $\cosh^{-1}(z)$, and $\tanh^{-1}(z)$.

   (c) When specific branches of the square root and the logarithmic functions are chosen, all these functions become single valued and analytic since they are then compositions of analytic functions. The derivatives are readily obtained from their logarithmic expressions, for example $\sin^{-1}(z)' = \frac{1}{(1-z^2)^{1/2}}$, $\tan^{-1}(z)' = \frac{1}{1+z^2}$, *etc.*

7. **Food for thought**: Think about the branch structure for more complicated functions like $\sqrt{(z-a)(z-b)}$.

1.3 Student Presentation: The Complex Infinity.

2 Reading assignment

Read Chapter 3 from the book.

3 Problem Set

Hand the following problems.

1. Find the real and imaginary parts of i^i, $\tan(\pi i)$, $\log(1)$, $\log(-1)$ and $\log(i)$.

2. Find all possible values of $\sin^{-1}(2)$.

3. Find all possible values of $i^{\sqrt{3}}$.

4. Prove that $(\tan^{-1}(z))' = \frac{1}{1+z^2}$ (find a logarithmic expression and specify a branch where this function is analytic first).

5. Determine all complex numbers z such that i^z has at most a finite number of values.

6. Use Euler's formula for the exponential and the well-known series expansions of the real functions e^x, $\sin(y)$, and $\cos(y)$ to show that
   $$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$
   (Hint: Use $(x+iy)^n = \sum_{j=0}^{n} \frac{n!}{j!(n-j)!} x^j (iy)^{n-j}$.)