Math 360 Complex Variables (Spring 2015) Week 6 Worksheet

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1 Week 6 Highlights

1.1 Unique Determination

A complex analytic function is *uniquely* determined by its values on a subdomain! (This is again different than the real case.)

1. If $f(z)$ is analytic throughout a domain D, and $f(z) = 0$ on a subdomain of D or on a line segment in D, then $f(z)$ is identically zero throughout D. Hence *the zeros of a nontrivial analytic function are isolated.*
2. An analytic function $f(z)$ is uniquely determined by its values on a subdomain or a line segment contained in D (if $g(z)$ is analytic and agrees with $f(z)$ on a subdomain or a line segment in D, let $h(z) = f(z) - g(z)$ and use (1) above to deduce $h = 0$ everywhere on D). The fact that a 'global' analytic function can be deduced from such few information illustrates the power of complex analyticity (diffrerentiability).

1.2 Analytic Continuation

Unique determination helps us study the questions: Can we extend the domain of an analytic function (analytic continuation) to a larger domain? What is the biggest possible domain of definition of a given analytic function (Riemann surfaces)?

1. $f_1(z)$ is an analytic continuation of $f(z)$ to a larger domain D_1 containing the domain D of f if $f_1(z)$ is analytic on D_1 and agrees with f on D.
2. An analytic continuation of an analytic function to a larger domain D_1 *may or may not* exist.
3. Whenever an analytic continuation exists, it is unique. (Careful though, if f_1 is the analytic continuation of f from D to D_1, f_2 is the analytic continuation of f_1 from D_1 to D_2, and D_2 happens to intersect D, then f_2 and f need not agree on that intersection.)
4. Sometimes, we have a formula for a function that is valid in a limited region of space (like series), and we ask whether we could find a representation (a closed form, series form, other forms) that is valid in a larger domain. This process of extending the domain of definition of an analytic function is analytic continuation. For example, the function $f(z) = \sum_{n=0}^{\infty} z^n$ which is analytic on the open unit disk $D = \{z : |z| < 1\}$, can be continued analytically to the whole complex plane except the point 1 ($\mathbb{C} \setminus \{1\}$) using the function $g(z) = \frac{1}{1-z}$, (which is clearly analytic over $\mathbb{C} \setminus \{1\}$, and agrees with f on D!).
5. There are some types of singularities that are so serious that they prevent analytic continuation of the function in question. Such singularities are called *natural barriers* (see for example problem 5 in this sheet). These are sometimes found in solutions of certain nonlinear differential equations arising in physical applications.

1.3 Reflection Principle (Schwarz)

This provides a way to extend the domain (analytic continuation) of a function which is analytic on the upper half plane and whose domain has a segment of the x-axis, to the entire plane by *reflection* (define $F(\bar{z}) = \overline{f(z)}$). The only requirement is that f is real when $z = x$ is real.

1. Reflection principle: If an analytic function has a *symmetric* domain about the x-axis, and that domain contains the x-axis, and the function is real when $z = x$ is real, then $f(\bar{z}) = \overline{f(z)}$.
2. Hence, if an analytic function maps the real line to the real line, then it must preserve symmetries with respect to the real line.
3. In particular, if an analytic function on the upper half plane maps the real axis to the real axis, then it can be continued analytically to the entire plane by defining $F(\bar{z}) = \overline{f(z)}$.
4. This can be generalized to preserving symmetries with respect to circles.

1.4 Student Presentation: Riemann Surfaces

2 Reading assignment

Read Sections 28 and 29 (pages 80- 85) from the book.

3 Problem Set

Hand the following problems.

1. List all the properties that you know about analytic functions so far.

2. Find all functions $f(z)$ satisfying all the following properties:

   (a) $f(z)$ is analytic on $\{\Im(z) > 0\}$,

   (b) $f(z)$ is continuous on $\{\Im(z) \geq 0\}$,

   (c) $f(z)$ is real on the real axis,

   (d) $|f(z)| > |\sin(z)|$ on $\{\Im(z) > 0\}$.

3. Find a function $f(z)$ that satisfies all the following properties:

   (a) $f(z)$ is analytic on $\{\Im(z) > 0\}$,

   (b) $f(z)$ is continuous on $\{\Im(z) \geq 0\}$ except at the origin,

   (c) $f(z)$ is real on the real axis (except at zero it is undefined),

   (d) $|f(z)| \leq \frac{c}{|z|^3}$ when $\{\Im(z) > 0\}$,

   (e) $f(i) = 4i$.

4. Use analytic continuation to find the value of the function $\sum_{n=1}^{\infty} \frac{z^n}{n}$ at $z = 3i$.

5. Consider the function defined by the power series $f(z) = \sum_{n=1}^{\infty} z^{n!}$. For what values of z does it converge? Is there an analytic continuation beyond the circle of convergence of the power series?