Math 360 Complex Variables (Spring 2015) Weeks 7 and 8 Worksheet

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1 Weeks 7 and 8 Highlights

1.1 Complex Integration

Just like in Calculus of real variables, we studied limits, continuity, differentiability of functions, and now we study integrability. In real Calculus, differentiation and integration are inverse operations to each other (Fundamental Theorem of Calculus). In complex analysis, this still holds, but we have more: integration and differentiation are equivalent, in the sense that we can obtain a function's derivatives by integrating it (with a weight around a contour). This is ***generalized Cauchy's integral formula*** (given below).

1.1.1 Integral of a complex function of a *real* variable $\int_a^b f(t)dt$

$$\int_a^b f(t)dt = \int_a^b u(t)dt + i\int_a^b v(t)dt.$$

1.1.2 Contour Integrals $\int_{z_0}^{z_1} f(z)dz$

Here, we need to consider a specific *path* Γ between z_0 and z_1, since there are many ways to connect these points in the complex plane. Then? *Parametrize* Γ by $z(t) = x(t) + iy(t)$ with $t \in [t_0, t_1]$, where $z(t_0) = z_0$, and $z(t_1) = z_1$. Now given that Γ is a smooth curve (has nonzero continuous derivative on $[t_1, t_2]$), and that $f(z)$ is continuous on Γ, then

$$\int_\Gamma f(z)dz = \int_{t_0}^{t_1} f(z(t))z'(t)dt.$$

1. The above integral is *independent* of the parametrization of the curve Γ. So we can always choose $t_0 = 0$ and $t_1 = 1$ and we can parametrize in $t \in [0, 1]$.

2. The parametrization gives the curve Γ a natural *direction* (put an arrow on the direction the curve is traversed as t increases from 0 to 1), so $-\Gamma$ will be the directed curve with the opposite direction, and $\int_{-\Gamma} f(z)dz = -\int_\Gamma f(z)dz$.

3. We can construct paths between z_0 and z_1 which are not smooth but which are piecewise smooth. Such paths are called ***contours***, and if $f(z)$ is continuous on the contour Γ which is made of n finitely many smooth curves Γ_k, then the *contour integral of* f along Γ is
$$\int_\Gamma f(z)dz = \sum_{k=1}^{n} \int_{\Gamma_k} f(z)dz.$$

4. Terminology and notation: A contour is ***closed*** if both its endpoints coincide. It is ***simple*** if it has no self-intersections. A closed simple contour is a ***loop***. A ***domain*** is an open subset of the complex plane which is (path) ***connected***, that is, every pair of points in the domain can be connected by a contour. A domain D is ***simply connected*** if the interior of every loop in D lies fully in D (for example, a punctured disk or an annulus are not simply connected). If Γ is a closed contour in a domain, we denote the integral along Γ by $\oint_\Gamma$, to emphasize the fact that the contour is closed.

5. ***Fundamental theorem of calculus*** for complex functions, contour integrals: If a complex function f is continuous on a domain D, and has an antiderivative F in D ($F'(z) = f(z)$ so F is analytic in D), Γ is any contour in D connecting z_0 to z_1, then $\int_\Gamma f(z)dz = F(z_1) - F(z_0)$. (So two things happen for a function that has an anti-derivative: Its integral does not depend on the path that we take from z_0 to z_1, only on the values of f at z_0 and z_1; if we integrate it along a closed contour, its integral will vanish).

6. ***Independence of path:*** Given f continuous on a domain D, the following three statements are *equivalent*:

   (a) f has an antiderivative F in D (so there is an F such that $F'(z) = f(z)$).

   (b) $\oint_\Gamma f(z)dz$ vanishes *for all* closed contours Γ in D.

   (c) The contour integrals $\int_\Gamma f(z)dz$ are independent of the path.

   Cauchy's integral theorem below tells us that if f is analytic, and our domain is simply connected, then the integral of f along *any* closed curve is zero (equivalently, f has an antiderivative on D)! Note that for independence of path, f need not be analytic, but must have an anti-derivative.

1.2 Cauchy's Integral Theorem

This is yet another powerful property of analytic functions, that has many important consequences. ***Statement***: If D is a *simply connected* domain and f is *analytic* on D, then for any *loop* Γ, $\oint_\Gamma f(z)dz = 0$.
Then from path independence above, we can conclude that an analytic function in a simply connected domain has an antiderivative, which is also analytic in D.

- Cauchy's integral theorem has this important consequence for computing contour integrals: *we can deform the contours into simpler ones*, and still get the same answer, as long as in doing so, we never cross a point at which f is not analytic. For example, instead of integrating $\oint_\Gamma \frac{1}{z}dz$ along an ellipse Γ enclosing the origin, we can integrate it along a circle C (deform the ellipse into a circle), and get the same answer ($2\pi i$).

- Using the above contour deformation, we see that for Γ any *positively oriented loop* in $\mathbb{C}$,

$$\oint_\Gamma \frac{1}{z - z_0}dz = 2\pi i,$$

  if z_0 lies in the *interior* of Γ and $\oint_\Gamma \frac{1}{z-z_0}dz = 0$ if z_0 lies in the exterior of Γ.

1.3 Cauchy's Integral Formula

This is powerful because it says that the values of an analytic function inside a simply connected domain (which is two dimensional) is determined by its values on the boundary (which is one dimensional)! Statement: If $f(z)$ is analytic in a simply connected domain D, and Γ is a positively oriented loop in D, and z_0 is in the interior of Γ, then

$$f(z_0) = \frac{1}{2\pi i}\oint_\Gamma \frac{f(\xi)}{\xi - z_0}d\xi.$$

1.4 Generalized Cauchy's Integral Formula

An analytic function is *infinitely differentiable*, and its derivatives are given by

$$f^{(n)}(z) = \frac{n!}{2\pi i}\oint_\Gamma \frac{f(\xi)}{(\xi - z)^{n+1}}d\xi.$$

Hence, the existence of one derivative of a complex functions implies the existence of infinitely many (not the case for real functions!). Moreover, taking complex derivatives is equivalent to evaluating integrals (also not the case for real functions). The above formula can be used to evaluate contour integrals, for example, if Γ is the unit circle, $\int_\Gamma \frac{e^z}{z^3} = \frac{2\pi i}{2!}(e^z)'' = \pi i$.

1.5 Cauchy's Estimate

This is an estimate for the derivatives of an analytic function. Let f be analytic in some domain D containing a circle C ($\{z : |z - z_0| = R\}$) of radius R centered around z_0. Suppose also that f is bounded on C, that is $|f(z)| < M$ for all z on the circle C, then

$$|f^{(n)}(z_0)| \leq \frac{n!M}{R^n}.$$

(Use generalized Cauchy's formula for the proof.)

1.6 Liouville's Theorem

This is an immediate consequence of Cauchy's estimate: *A bounded entire function is constant.* Recall that the main entire functions that we have encountered so far are polynomial and exponential functions, and some multiplications and compositions of these, which are clearly not bounded.
We can use Liouville's theorem to easily prove ***the fundamental theorem of algebra:*** Every non constant polynomial has at least one zero (more generally, counting multiplicity, a polynomial of degree n has n zeros in $\mathbb{C}$).

1.7 Morera's Theorem

This is the converse of Cauchy's integral theorem. If $f(z)$ is continuous in D and all loop integrals of $f(z)$ in D vanish, then f is analytic! (This is mainly due to the fact that if f is continuous with an antiderivative, then f is analytic, by making use of generalized Cauchy's integral formula.)

1.8 Maximum Modulus Principle

Let f be an analytic function which is not constant on a domain D, then $|f(z)|$ has no maximum value in D.
From this it follows that if f is continuous on a *closed and bounded* region D and is analytic and not constant in the interior of D, then $|f(z)|$ has a maximum (which is always attained) on the *boundary* of D. (Compare to the maximum principle for harmonic functions.) In Problem 3, we prove that $|f(z)|$ also attains its minimum on the boundary of D (given $|f(z)| \neq 0$).

2 Reading assignment

Read Chapter 4 from the book.

3 Problem Set

Hand the following problems.

1. Evaluate $\int_C z^n dz$ for integer n and some simple closed contour C that encloses the origin (consider the cases $n \neq -1$ and $n = -1$ separately) by:

   (a) Parametrizing C.

   (b) Finding an antiderivative and using the Fundamental Theorem of Calculus.

2. Consider the integral $I = \int_{-\infty}^{\infty} \frac{dx}{x^2+1}$.

   (a) Evaluate the integral using real calculus methods (this one is easy, but other real integrals will be very difficult without using complex techniques).

   (b) Evaluate the integral by considering $\oint_C \frac{dz}{z^2+1}$ where C is the closed semicircle in the upper half plane with endpoints at $(-R, 0)$ and $(R, 0)$ plus the x-axis. (Hint: use $\frac{1}{z^2+1} = \frac{-1}{2i}\left(\frac{1}{z+i} - \frac{1}{z-i}\right)$, and show that the integral along the open semicircle in the upper half plane vanishes as $R \to \infty$.)

3. Let f be continuous on a closed and bounded region D, and let it be analytic and non constant throughout the interior of D. Assuming that $f(z) \neq 0$ anywhere in D, prove that $|f(z|$ has a minimum value in D which occurs at the boundary of D and never in the interior. (Hint: Apply the maximum modulus principle for the function $g(z) = 1/f(z)$.)

4. Let $f(z)$ be an entire function, and let $M(R) = \max_{|z|=R} |f(z)|$ for $R > 0$. Suppose that $M(2R) < 2^N M(R)$ for all $R > 0$ and for some integer N. Show that $f(z)$ is a polynomial of degree not exceeding N. (Hint: Prove that the Taylor coefficient $f^{(N+k)}(0) = 0$ by making use of Cauchy's estimate, and sending R to infinity since the function is entire.)

5. Suppose that $f(z)$ is entire and has n simple zeros at $z_1, z_2, \ldots, z_n$. Suppose also that $|f(z)| \leq k|z|^m + L$ for some m, k and L. What is $f(z)$? (Hint: Like in problem 4 use Cauchy's estimate and let $R \to \infty$ since the function is entire.)

6. From the book: page 132 numbers 1(b), 2, 5.

7. From the book: page 170 numbers 1(b), 2(b), 4.

8. From the book: page 177 number 4.

4 Student Presentation: Generating Fractals Using Complex Functions

Explain the methodology and Download Fractint to generate the pictures.