The Structure of Chaos

How the Geometry of the Phase Space Influences a System's Dynamics

Ben Dulaney

MATH 441, April 28, 2016

Deterministic Systems

- For any deterministic system, given some initial conditions and differential equations governing its behaviour, we can predict the system's future.

Deterministic Systems

- For any deterministic system, given some initial conditions and differential equations governing its behaviour, we can predict the system's future.
- But we can't measure the positions and velocities of objects with absolute precision.

Deterministic Systems

- For any deterministic system, given some initial conditions and differential equations governing its behaviour, we can predict the system's future.
- But we can't measure the positions and velocities of objects with absolute precision.
- The discrepancy between the actual value of a measurement and its approximation can grow rapidly as time increases, giving us false predictions.

Deterministic Systems

- For any deterministic system, given some initial conditions and differential equations governing its behaviour, we can predict the system's future.
- But we can't measure the positions and velocities of objects with absolute precision.
- The discrepancy between the actual value of a measurement and its approximation can grow rapidly as time increases, giving us false predictions.

$$||\delta(t)|| \sim ||\delta_0||e^{\lambda t}$$

Attractors

- An *attractor* A is a closed set with the following properties:

Attractors

- An *attractor* A is a closed set with the following properties:
- 1. A is an invariant set: any trajectory $\mathbf{x}(t)$ that starts in A stays in A for all time.

Attractors

- An *attractor* A is a closed set with the following properties:
- 1. A is an invariant set: any trajectory $\mathbf{x}(t)$ that starts in A stays in A for all time.
- 2. A attracts an open set of initial conditions: there is an open set U containing A such that if $\mathbf{x}(0) \in U$, then the distance from $\mathbf{x}(t)$ to A tends to 0 as $t \to \infty$. The largest such U is called the basin of attraction of A.

Attractors

- An *attractor* A is a closed set with the following properties:
- 1. A is an invariant set: any trajectory $\mathbf{x}(t)$ that starts in A stays in A for all time.
- 2. A attracts an open set of initial conditions: there is an open set U containing A such that if $\mathbf{x}(0) \in U$, then the distance from $\mathbf{x}(t)$ to A tends to 0 as $t \to \infty$. The largest such U is called the basin of attraction of A.
- 3. A is minimal: there is no proper subset of A satisfying conditions 1 and 2.

Attractors

- An *attractor* A is a closed set with the following properties:
- 1. A is an invariant set: any trajectory $\mathbf{x}(t)$ that starts in A stays in A for all time.
- 2. A attracts an open set of initial conditions: there is an open set U containing A such that if $\mathbf{x}(0) \in U$, then the distance from $\mathbf{x}(t)$ to A tends to 0 as $t \to \infty$. The largest such U is called the basin of attraction of A.
- 3. A is minimal: there is no proper subset of A satisfying conditions 1 and 2.

Strange Attractor

A *strange attractor* is defined to be an attractor that demonstrates sensitive dependence on initial conditions.

Rossler System

$$x' = -y - z$$
$$y' = x + ay$$
$$z' = b + z(x - c)$$

Rossler System

$$x' = -y - z$$
$$y' = x + ay$$
$$z' = b + z(x - c)$$

- There are two equilibrium points $E_{\pm}$ for this system located at $(x_{\pm}, y_{\pm}, z_{\pm}) = (\frac{c \pm \sqrt{c^2 - 4ab}}{2}, -\frac{c \pm \sqrt{c^2 - 4ab}}{2a}, \frac{c \pm \sqrt{c^2 - 4ab}}{2a})$.

Rossler System

$$x' = -y - z$$
$$y' = x + ay$$
$$z' = b + z(x - c)$$

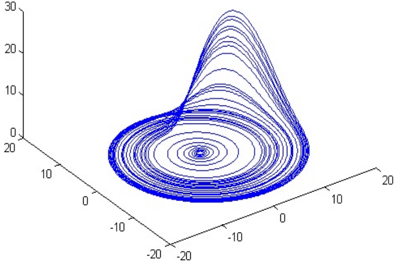
- There are two equilibrium points $E_{\pm}$ for this system located at $(x_{\pm}, y_{\pm}, z_{\pm}) = (\frac{c \pm \sqrt{c^2 - 4ab}}{2}, -\frac{c \pm \sqrt{c^2 - 4ab}}{2a}, \frac{c \pm \sqrt{c^2 - 4ab}}{2a})$.
- E_{-}, a *saddle-focus*, is always unstable, so a trajectory near it spirals outward mainly in the x-y plane

Rossler System

$$x' = -y - z$$
$$y' = x + ay$$
$$z' = b + z(x - c)$$

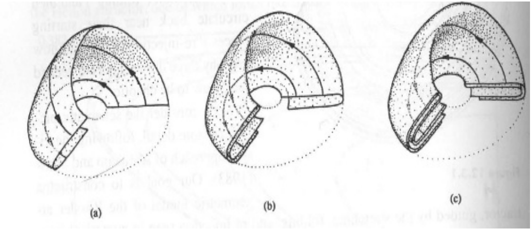
- There are two equilibrium points $E_\pm$ for this system located at $(x_\pm, y_\pm, z_\pm) = (\frac{c \pm \sqrt{c^2 - 4ab}}{2}, -\frac{c \pm \sqrt{c^2 - 4ab}}{2a}, \frac{c \pm \sqrt{c^2 - 4ab}}{2a})$.
- E_-, a *saddle-focus*, is always unstable, so a trajectory near it spirals outward mainly in the x-y plane
- E_+, also an unstable saddle-focus, is located outside the attractor. When a trajectory gets sufficiently close to E_+ it gets pushed back down to the x-y plane, and a new cycle occurs.

Rossler Attractor

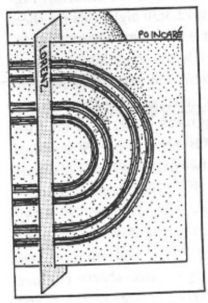


stretch $\rightarrow$ fold $\rightarrow$ re-inject

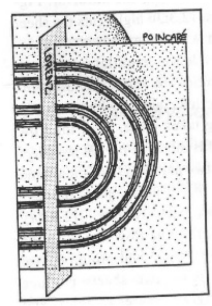
Paper-Sheet Model



Poincare Section

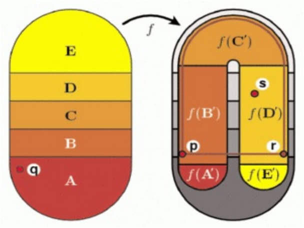


Poincare Section



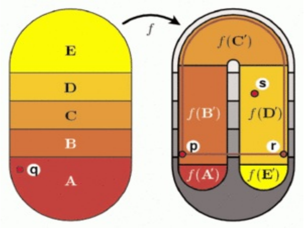
- This model of the Rossler attractor is locally the Cartesian product of the *Cantor set* and a 2D *manifold* (ribbon).

Smale Horseshoe Map



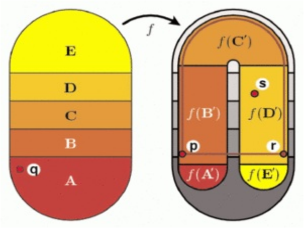
- The semi-disks A and E are contracted to the semi-disks $f(A)$ and $f(E)$, both in A.
- B and D are sent linearly to $f(B)$ and $f(D)$ through a vertical stretch and a horizontal shrink.
- C gets mapped to $f(C)$ in the region E.

Smale Horseshoe Points



- The point q is a sink (stable fixed point) since for any point $z = (x, y) \in A \cup E \cup C$ we have that z converges to q under the forward iteration, $f^n(z) \to q$ as $n \to \infty$.

Smale Horseshoe Points



- The points p and s are saddle points.
- If z lies on the horizontal line passing through p, then $f^n(z) \to p$ as $n \to \infty$.
- While if z lies on the vertical line going through p, then $f^n(z) \to p$ as $n \to -\infty$.

Smale Horseshoe

- We can now define two sets,

$$W^s = \{z \mid f^n(z) \to p \text{ as } n \to \infty\}$$

$$W^u = \{z \mid f^n(z) \to p \text{ as } n \to -\infty\}$$

which represent the invariant stable and unstable manifolds of p respectively.

- The transverse intersection of these two manifolds occurs at r, and so r is called a transverse *homoclinic point*.

Smale Horseshoe Dynamics

- The important part of the horseshoe dynamics happen on the set,
$$\Lambda = \{z \mid f^n(z) \in B \cup D \; \forall n \in \mathbb{Z}\}.$$

Smale Horseshoe Dynamics

- The important part of the horseshoe dynamics happen on the set,
$$\Lambda = \{z \mid f^n(z) \in B \cup D \ \forall n \in \mathbb{Z}\}.$$
- Take two symbols, 0 and 1. Let Σ be set of all bi-infinite sequences $a = \{a_n\}, n \in \mathbb{Z}$ such that $a_n = 0$ or $a_n = 1$ for each $n \Rightarrow \Sigma = \{0, 1\}^{\mathbb{Z}}$.
- We next define the homeomorphic function $\sigma : \Sigma \to \Sigma$ by $\sigma(a) = a_{n+1}$.

Smale Horseshoe Dynamics

- It can be shown there also exists a homeomorphism $h : \Sigma \to f_\Lambda$
- So, given any $a \in \Sigma$ there is a unique $z \in \Lambda$ such that $f^n(z) \in B$ whenever $a_n = 1$, while $f^n(z) \in D$ whenever $a_n = 0$.
- Thus, σ codes the horseshoe dynamics.

Smale Horseshoe Dynamics

- It can be shown there also exists a homeomorphism $h : \Sigma \to f_\Lambda$
- So, given any $a \in \Sigma$ there is a unique $z \in \Lambda$ such that $f^n(z) \in B$ whenever $a_n = 1$, while $f^n(z) \in D$ whenever $a_n = 0$.
- Thus, σ codes the horseshoe dynamics.
- Every dynamical property of the shift map σ is possessed equally by f_Λ.
- For example, Since σ has 2^n periodic orbits of length n, so does f_Λ.

Smale Horseshoe Chaos

- And, so small changes in the initial conditions of Σ can produce significant changes in a σ orbit, so this must also be true for f_Λ.

Smale Horseshoe Chaos

- And, so small changes in the initial conditions of Σ can produce significant changes in a σ orbit, so this must also be true for f_Λ.
- Therefore, the chaos of σ is reproduced exactly in Smale's horseshoe!

Smale Horseshoe Chaos

- And, so small changes in the initial conditions of Σ can produce significant changes in a σ orbit, so this must also be true for f_Λ.
- Therefore, the chaos of σ is reproduced exactly in Smale's horseshoe!

We Showed:

transverse homoclinicity $\Rightarrow$

Smale Horseshoe Chaos

- And, so small changes in the initial conditions of Σ can produce significant changes in a σ orbit, so this must also be true for f_Λ.
- Therefore, the chaos of σ is reproduced exactly in Smale's horseshoe!

We Showed:

transverse homoclinicity $\Rightarrow$ horseshoe $\Rightarrow$

Smale Horseshoe Chaos

- And, so small changes in the initial conditions of Σ can produce significant changes in a σ orbit, so this must also be true for f_Λ.
- Therefore, the chaos of σ is reproduced exactly in Smale's horseshoe!

We Showed:

transverse homoclinicity $\Rightarrow$ horseshoe $\Rightarrow$ chaos

Fin

" *Geometry is not true, it is advantageous*"
-Henri Poincare