The Logistic Map and Chaos

Dylana Wilhelm

James Madison University

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Logistic Map

The logistic map is a first-order difference equation discovered to have complicated dynamics by mathematical biologist Robert May. The general form is given by

$$x_{n+1} = rx_n(1 - x_n),$$

where x_n is the population of nth generation and $r \geq 0$ is the growth rate.

Applications of the Logistic Map

- Generally used in population biology to map the population at any time step to its values at the next time step
- Additional applications include:
  - Genetics - change in gene frequency
  - Epidemiology - fraction of population infected
  - Economics - relationship between commodity quantity and price
  - Social Sciences - number of people to have heard a rumor

Logistic Map Derivation

- Derived from the logistic difference equation

$$N_{n+1} = N_n(r - aN_n),$$

  by letting $x = aN/r$
- Results in simplest non-linear difference equation

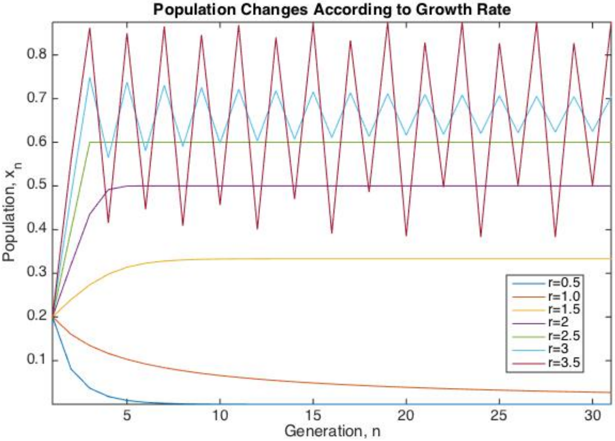
$$x_{n+1} = rx_n(1 - x_n)$$

- Parabola with a maximum value of $r/4$ at $x_n = 1/2$
- For $0 \leq r \leq 4$, maps $0 \leq x_n \leq 1$ into itself

Variation of Growth Rate

- Limit growth rate to interval $0 \leq r \leq 4$
- Range of behavior as r is varied:
  - Population reaches extinction for $r < 1$
  - Non-trivial steady state for $1 < r < 3$
  - Fluctuations in population for $r > 3$

Variation of Growth Rate



Fixed Points of Logistic Map

Fixed points satisfy the equation

$$f(x^*) = x^* = rx^*(1 - x^*),$$

which gives

$$x^*(1 - r + rx^*) = 0.$$

Thus, we have that

$$x^* = 0 \quad \text{or} \quad x^* = 1 - \frac{1}{r}.$$

Stability of Fixed Points

Stability is given by $|f'(x^*)| < 1$. We have that $f'(x^*) = r - 2rx^*$. Thus,

- $|f'(0)| = |r| \implies x^* = 0$ is stable for $r < 1$
- $|f'(1 - \frac{1}{r})| = |2 - r| \implies x^* = 1 - \frac{1}{r}$ is stable for $1 < r < 3$

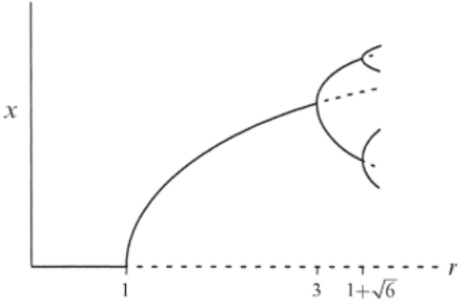
A period-2 cycle exists for $r > 3$. A fixed point p is stable if $|\frac{d}{dx} f(f(p))| < 1$. Therefore,

- the 2-cycle is stable for $3 < r < 1 + \sqrt{6}$

Bifurcations

- For $r < 1$, $x^* = 0$ is stable.
- At $r = 1$, a transcritical bifurcation occurs and $x^* = 0$ becomes unstable.
- For $1 < r < 3$, $x^* = 1 - \frac{1}{r}$ is stable.
- At $r = 3$, a flip bifurcation occurs and $x^* = 1 - \frac{1}{r}$ becomes unstable.
- A period-2 cycle is stable for $3 < r < 1 + \sqrt{6}$.

Bifurcation Diagram



- Solid line indicates stable
- Dashed line indicates unstable

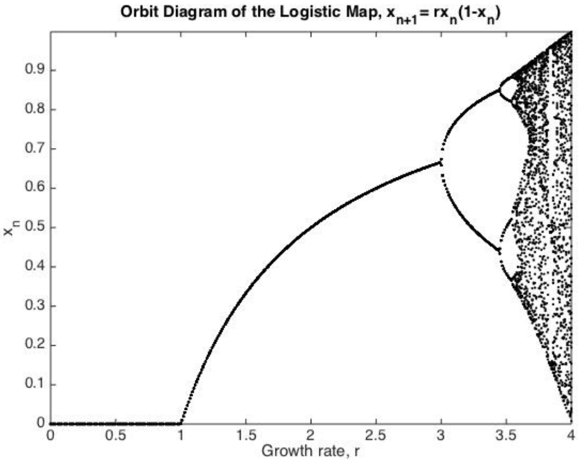
Period Doubling

As the period-2 cycle becomes unstable, a stable period-4 cycle appears. The period-4 cycle becomes unstable with the emergence of a stable period-8 cycle. This is followed by a period-16 cycle and so on.

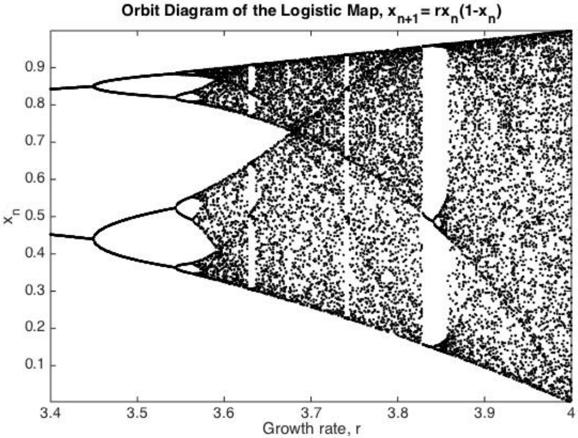
The parameter value r_n at which a period-2^n cycle is created converges geometrically to a limiting value r_∞. The distance between successive transitions is given by

$$\delta = \lim_{n \to \infty} \frac{r_n - r_{n-1}}{r_{n+1} - r_n} = 4.6692016091...$$

Orbit Diagram

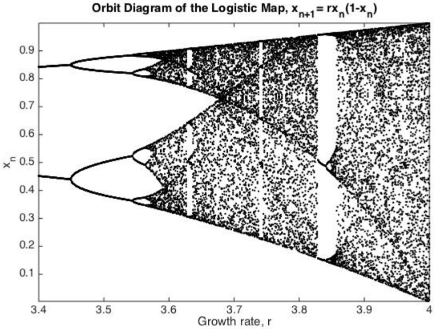


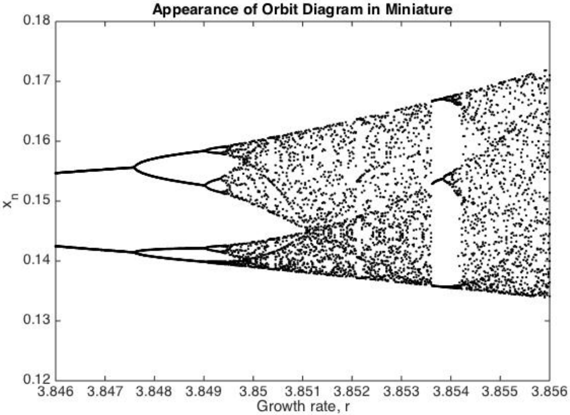
Period Doubling



Chaotic Behavior & Period Windows

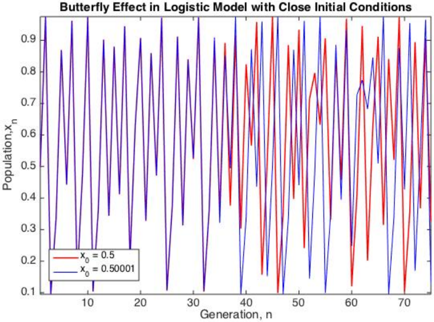
- At $r = r_\infty \approx 3.57$, map becomes chaotic and settles toward an infinite number of values.
- Periodic windows emerge out of chaotic behavior
- Beginning of period-3 window at $r \approx 3.83$ defined by a tangent bifurcation.





Same pattern exists at every scale - zooming into the orbit diagram exposes smaller copies of the larger structure

Butterfly Effect



- Discovered by Lorenz
- Arbitrarily close initial conditions can lead to trajectories that diverge over time

Conclusion

- Simple first-order nonlinear equation with extraordinary dynamical behavior
- Presents transition from stable fixed points to stable cycles to fluctuations between infinite values
- Presence of fractal form
- Divergent behavior over time despite equation's deterministic simplicity
- Sensitive dependence on initial conditions

References

- Boeing, Geoff. "Chaos Theory and the Logistic Map." Web.
- May, Robert M. "Simple mathematical models with very complicated dynamics." *Nature* (1976) 456-67.
- Strogatz, Stephen H. *Nonlinear Dynamics and Chaos*. 2nd edition. 2015: Westview Press. 360-73.
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