Navier-Stokes Equation
Derivation and Applications

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Equation and Key assumptions for this equation

Navier- Stokes equation

Incompressible - Density remains constant throughout the fluid

Newtonian - Viscous stresses are linearly proportional to the strain rate

Conservation of Momentum is upheld - Total momentum is constant throughout the system

Reynolds Transport Theorem

$$\frac{d}{dt}\int_{\Omega} LdV = -\int_{\partial\Omega} L\vec{v}\cdot\vec{n}dA - \int_{\Omega} QdV$$

$$\int_{\partial\Omega} L\vec{v}\cdot\vec{n}dA = \int_{\Omega} \nabla\cdot(L\vec{(v)})dV$$

Left equation is the change of a property contained within a volume

first right equation is input/output of the property from the boundaries

final term refers to how much is being input/output from within the volume (i.e. sinks and sources)

Conservation of Momentum (Material Derivative)

Newtons 2nd law states F = ma

Using $\vec{b}$ instead of mass you get $\rho \frac{d}{dt} \vec{v}(x, y, z, t)$

By chain rule $\vec{b} = \rho(\frac{\partial \vec{v}}{\partial t} + \frac{\partial \vec{v}}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial \vec{v}}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial \vec{v}}{\partial z}\frac{\partial z}{\partial t}) = \rho(\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v})$

This is the material derivative $\frac{D\vec{v}}{Dt}$

Derivation

begin with $\vec{b} = \nabla \cdot \sigma + \vec{f}$ with σ defined as below

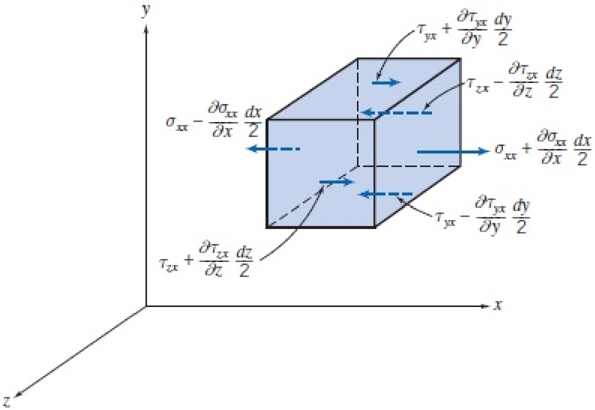
$$\begin{pmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{pmatrix}$$

Can rearrange to look like

$$\begin{pmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{pmatrix} + \begin{pmatrix} \sigma_{xx} + p & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} + p & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} + p \end{pmatrix}$$

Physics of the Navier-Stokes Equation

$$\frac{D\vec{V}}{Dt} = -\frac{1}{\rho}\nabla p + \frac{\mu}{\rho}\nabla^2\vec{v} + \vec{g}$$



Non-Dimensionalizing Navier-Stokes

$$\frac{D\vec{V}}{Dt} = -\frac{1}{\rho}\nabla p + \frac{\mu}{\rho}\nabla^2 \vec{v} + \vec{g}$$

Length $r^* = \frac{r}{L}$ Velocity $\vec{v}^* = \frac{\vec{v}}{U}$ Time $t^* = \frac{t}{L/U}$

Pressure $p^* = \frac{p}{\rho U^2}$ or $p^* = \frac{pL}{\mu U}$

Discussion of Froude and Reynolds number

$$\frac{\partial \vec{v}^*}{\partial t^*} + (\vec{v}^* \cdot \nabla)\vec{v}^* = -\nabla p^* + \frac{1}{Re}\nabla^2 \vec{v}^* + \frac{1}{Fr^2}\hat{g}$$

$$Re = \frac{\rho U L}{\mu} \;\; Fr = \sqrt{\frac{U}{gL}}$$

Re number is used to discuss whether viscous forces are the most dominant, or whether inertial forces are the most dominant (Smooth v. Chaotic)
Fr number is used to discuss differences in wave patterns between models (i.e used for scaling)

Millennium Prize Problem

Problem: Prove that in spacetime given an initial vector field, there exists a vector velocity and scalar pressure field, which are both smooth and globally defined which solve the NS equation.
The Navier-Stokes equations have been proven smooth and globally defined in 2D