# Chaos and Strange Attractors

SHICHENG LIU

# Definitions

- Chaos --- study of dynamical systems (non-periodic systems in motion) usually over time, it is unpredictable behavior of deterministic system.

- Attractor---a set of points in phase space toward which neighboring points gradually approach within a basin of attraction.

  ---An attractor in phase space, where the points never repeat themselves, and orbits never intersect, but they stay within the same region of phase space is called strange attractor.

# Definitions and Application of Attractors

- Strange Attractor---an attractor that exhibits sensitive dependence on initial conditions; An attractor can be a point, curve, or space. They usually have fractal structure (infinite detail).

---There are several types of strange attractors including:

- Chua—used in electronic circuitry.
- Duffing—used in nonlinear oscillators.
- Rössler—used in chemical kinetics.
- Ikeda—involved in the turbulence of trails of smoke.
- Lorenz—used in atmospheric convection.

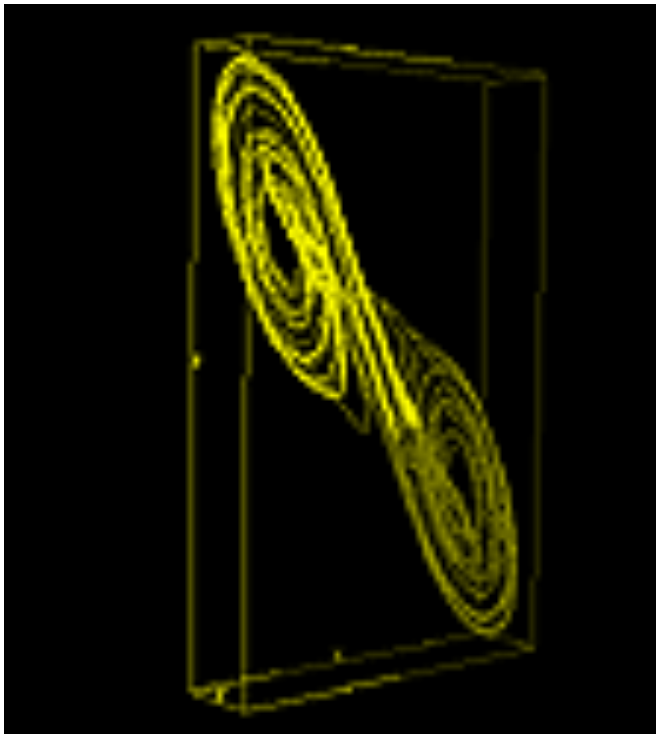
# Chua

- Chua's circuit is a simple electronic circuit that exhibits classic chaos theory behavior, which roughly means "nonperiodic oscillator"; it produces an oscillating waveform that never "repeats". The Chua's attractor, known as "The Double Scroll".
- The function f(x) describes the electrical response of the nonlinear resistor, and its shape depends on the particular configuration of its components. The parameters α and β are determined by the particular values components.

$$\frac{dx}{dt} = \alpha[y - x - f(x)]$$

$$RC_2\frac{dy}{dt} = x - y + Rz$$

$$\frac{dz}{dt} = -\beta y$$

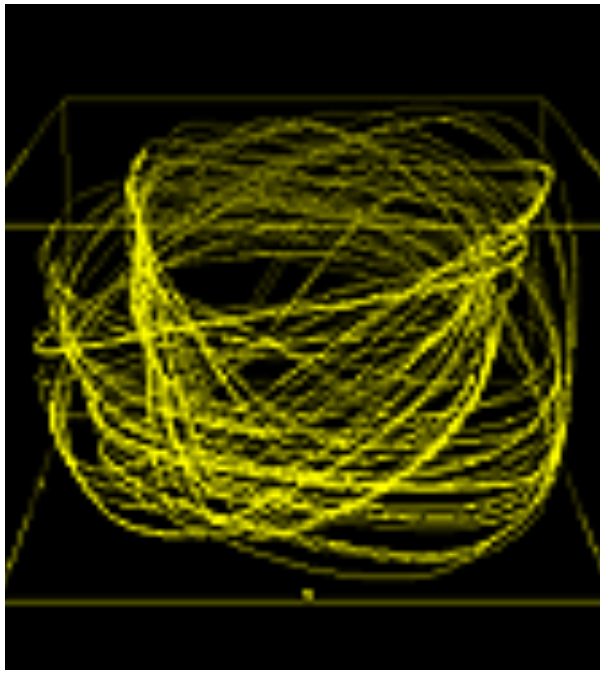


# Duffing

- The Duffing oscillator is used to model certain damped and driven oscillator.
- The Duffing equation describes the motion of a damped oscillator with a more complex potential than in simple harmonic motion. For example, it model a spring pendulum whose spring's stiffness doesn't exactly obey Hooke's law.

$$\ddot{x} + \delta\dot{x} + \beta x + \alpha x^3 = \gamma \cos \omega t \ ,$$

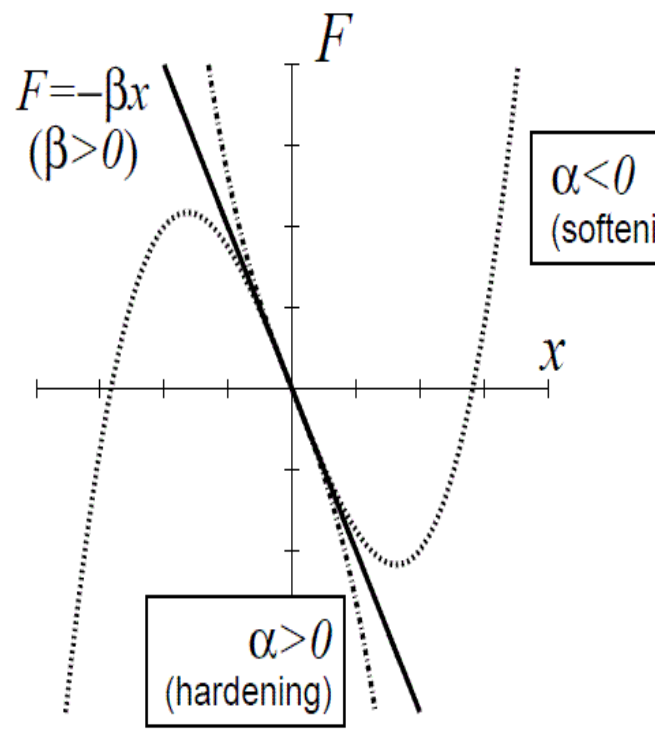
- Δ controls the amount of damping constant . α controls the linear stiffness. β controls the amount of non-linearity in the restoring force. Γ is the amplitude of the periodic driving force. Ω is the angular frequency of the periodic driving force.



# Duffing(cont.)

- For β>0 , the Duffing oscillator can be interpreted as a forced oscillator with a spring whose restoring force is written as F=−βx−αx^3. When α>0 , this spring is called a hardening spring, and, when α<0 , it is called a softening spring.

- For β<0 , the Duffing oscillator describes the dynamics of a point mass in a double well potential, and it can be regarded as a model of a periodically forced steel beam which is deflected toward the two magnets.



# Rössler

- The Rössler attractor is the solution of these 3 non-linear differential equations:

  $dx/dt = -y - z$

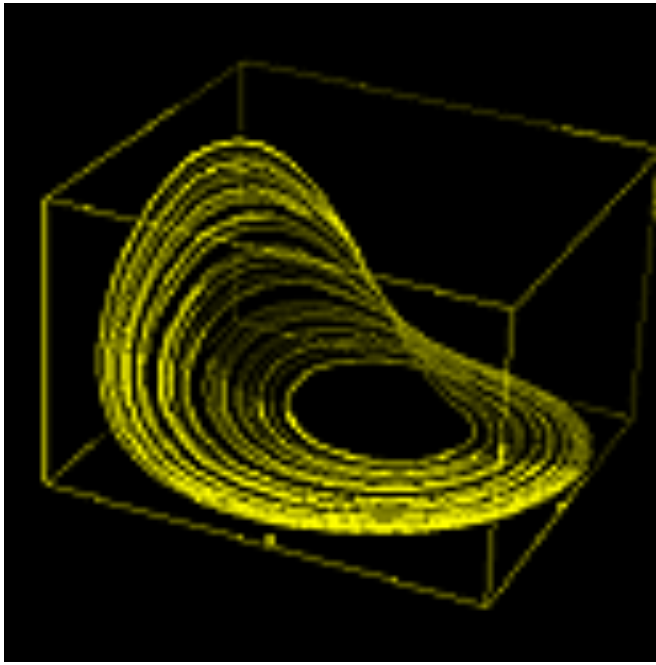
  $dy/dt = x + ay$

  $dz/dt = b + z(x - c)$

  where $a=0.2$, $b=0.2$, $c=5.7$

- An orbit within the attractor follows an outward spiral close to the X, Y plane around an unstable fixed point. Once the graph spirals out enough, a second fixed point causes a rise and twist in the Z dimension.

- The series does not form limit cycles nor does it ever reach a stable state. Instead it is an example of deterministic chaos. As with other chaotic systems the Rössler system is sensitive to the initial conditions.



# Ikeda

- The Ikeda map is a discrete-time dynamical system given by the complex map:

$$z_{n+1} = A + Bz_n e^{i(|z_n|^2 + C)}$$

- Zn stands for the electric field inside the resonator at the n-th step of rotation in the resonator, A and C are parameter which indicates laser light applied from the outside, and linear phase across the resonator, respectively.
- In particular, the parameter B<1 is called dissipation parameter characterizing the loss of resonator, and in the limit of B=1 the Ikeda map becomes a conservative map.
- A 2D real example of the above form is:

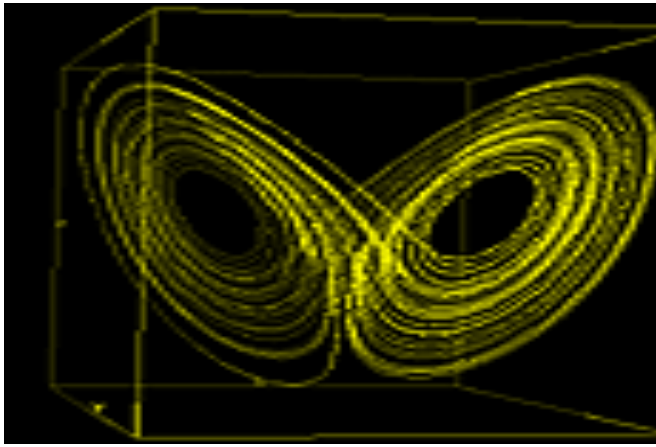
$$x_{n+1} = 1 + u(x_n \cos t_n - y_n \sin t_n),$$

$$y_{n+1} = u(x_n \sin t_n + y_n \cos t_n),$$

- For u>= 0.6, this system has a chaotic attractor.

# Lorenz

- Lorenz original derivation of these equations are from a model for fluid flow of the atmosphere: a two-dimensional fluid cell is warmed from below and cooled from above.
- By rising the temperature difference between the two surfaces, we observe initially a linear temperature gradient. The resulting convective motion is modeled by a partial differential equation.
- X measures the velocity field. Y measures the horizontal temperature variation. Z measures the vertical temperature variation . σ depends on the type of fluid, which is called Prandtl number. b depends on the geometry, that is related to the physical size of the system. r depends on the Rayleigh number, which influences the change of behavior from conductive to convective.



$$\frac{dx}{dt} = \sigma(y - x)$$

$$\frac{dy}{dt} = -xz + rx - y$$

$$\frac{dz}{dt} = xy - bz$$

# Conclusion

- Chaos theory do give us the tools to approximate real systems by mathematics and to tell us how good those approximations are. For instance, by running weather models using slightly different initial conditions forecasters can tell how predictable the weather will be. If the results diverge widely between the two sets of data, then the weather is in an unpredictable mode, otherwise they know their forecasts will be pretty good.

- Chaos Theory challenges old assumptions that most things are predictable by math and physics. The theory covers many areas but primarily dynamic systems and how they operate. It describes that almost all things in life follow a dynamic pattern, not a linear one, even though math, physics, engineering are based on linear systems.