Math 441 Final Exam Spring 2016
Name:

Answer the following to the best of your knowledge.

1. (Gradient system)

   (a) Show that closed orbits are impossible in gradient systems $\dot{x}(t) = -\nabla V(x(t))$.

   (b) Show that the following system is a gradient system and find its potential,

   $$\begin{cases} \dot{x}(t) = y + 2xy, \\ \dot{y}(t) = x + x^2 - y^2. \end{cases}$$

2. (Linear stability) For a two dimensional flow,

   $$\begin{cases} \dot{x}(t) = f(x,y), \\ \dot{y}(t) = g(x,y), \end{cases}$$

   with an isolated fixed point (x^*, y^*), show that the linearized system is $\dot{\vec{x}}(t) = J(x^*, y^*)\vec{x}(t)$, where $J(x^*, y^*)$ is the Jacobian matrix of $f(x,y)$ and $g(x,y)$ evaluated at (x^*, y^*). What happens in the case $J(x^*, y^*) = 0$?

3. (Theory) State Poincare Bendixson theorem for two dimensional flows.

4. (ω-limit sets) Define the ω-limit set of $\omega(x)$ where $x \in \mathbb{R}^n$ is a phase point, then state all its properties that you can recall.

5. (Limit cycles) Use Poincare Bendixson theorem to show that the following system has a limit cycle, and give tight bounds for the radius of the limit cycle,

   $$\begin{cases} \dot{r}(t) = r(1 - r^2) + \mu r\cos(\theta), \\ \dot{\theta}(t) = 1. \end{cases}$$

6. (Hopf bifurcation)

   (a) Show that the following system exhibits a super-critical Hopf bifurcation,

   $$\begin{cases} \dot{r}(t) = \mu r - r^3, \\ \dot{\theta}(t) = \omega + br^2. \end{cases}$$

   (b) Express the system in cartesian coordinates, linearize about the origin, and describe the stability and type of this fixed point for various values of μ. What does the parameter ω control?

7. (Fractals) Compute the fractal dimension of the Sierpinski triangle (constructed as follows: Start with an equilateral triangle, subdivide into four smaller congruent equilateral triangles and remove the central one, repeat with each of the remaining ones).

8. (Vocabulary) Define strange attractor.

9. (Lorentz system) Consider the Lorentz equations

   $$\begin{cases} x' = \sigma(y - x) \\ y' = rx - y - xz \\ z' = xy - bz, \end{cases}$$

   where σ, r, and b are constant parameters.

   (a) (Global stability of the origin) Using the potential function $V(x,y,z) = \frac{1}{\sigma}x^2 + y^2 + z^2$, show that the origin is globally stable when $0 < r < 1$.

   (b) Show that for the Lorenz system, the volumes in the phase space shrink exponentially fast (Hint: A volume $V(t)$ enclosed by a closed surface $S(t)$ in phase space evolves with $\dot{\vec{x}}(t) = \vec{f}(\vec{x})$ as $V'(t) = \int_S \vec{f}\vec{n}dA$, then use the divergence theorem). Does this mean that the ω-limit set is a point?

10. (Overview) What do you feel you learned the most in this class? What's the best thing you learned?

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