Math 441 (Spring 2017) Final Exam

Name:

Attempt all questions to the best of your knowledge

1. (**Picard's iteration**) Prove, using Picard's iteration, that the solution to the system $y' = 2y$, $y(0) = 1$ is the function $y(t) = e^{2t}$.

2. (**Existence and uniqueness**) Write a precise statement of Picard's existence and uniqueness theorem for the system $x' = f(t, x)$, $x(0) = x_0$, then prove the existence part.

   What happens if you drop the Lipschitz condition on f?

3. (**Alpha and Omega limit sets**)

   (a) Define an ω-limit set for a dynamical system $\vec{x}' = \vec{f}(\vec{x})$.

   (b) Prove that the ω-limit set is closed (it contains all its limit points).

4. (**Poincare Bendixson Theorem**)

   (a) Why does Poincare Bendixon Theorem only work for two dimensional dynamical systems? (What would go wrong in the proof had the system been higher dimensional?)

   (b) Prove that the system $r' = r(1 - r^2) + \mu r \cos(\theta)$, $\theta' = 1$ has a limit cycle in the region bounded between $0.999\sqrt{1-\mu} < r < 1.001\sqrt{1+\mu}$.

5. (**Hamiltonian system**) Consider the dynamical system, $x' = \frac{\partial H}{\partial y} + \mu H \frac{\partial H}{\partial x}$, $y' = -\frac{\partial H}{\partial x} + \mu H \frac{\partial H}{\partial y}$, where $H(x,y) = \frac{1}{2}y^2 - \frac{1}{2}x^2 + \frac{1}{4}x^4$.

   (a) Prove that $H' = \frac{dH}{dt} = \mu H \left[\left(\frac{\partial H}{\partial x}\right)^2 + \left(\frac{\partial H}{\partial y}\right)^2 \right]$.

   (b) Deduce that $H \to 0$ as $t \to \infty$ if $\mu < 0$.

   (c) Sketch the graph of $H(x,y) = 0$ (*hint*: infinity figure).

   (d) Deduce that any orbit starting outside the set $H = 0$ will approach it from the outside. What is the ω-limit set of such an orbit?

   (e) Make a sketch of the phase portrait.

6. (**Gradient system**) Prove that gradient systems cannot have limit cycles.

7. (**Lyapunov function**) Find a and b so that the function $V = ax^2 + by^2$ is a Lyapunov function for the system $x' = y - x^3$, $y' = -x - y^3$.

8. (**Gronwal's inequality**) Prove Gronwal's inequality: Let $u : [0, \alpha] \to \mathbb{R}$ be continuous and nonnegative. Suppose $C \geq 0$ and $K \geq 0$ are such that $u(t) \leq C + \int_0^t Ku(s)ds$ for all $t \in [0, \alpha]$. Then for all $t \in [0, \alpha]$, $u(t) \leq Ce^{Kt}$.

9. (**Chaos**)

   (a) What makes a dynamical system chaotic?

   (b) Can a linear system be chaotic?

   (c) Define the term *strange attractor*. Contrast with a 'non-strange' attractor.