## Math 441 (Spring 2017) Assignment Two

1. (Taylor expansion, graphics, smooth and analytic functions)
(a) Explain the difference between smooth and analytic real functions.
(b) Prove that the function $\frac{\sin x}{x}$ is analytic on $\mathbb{R}$ and find its Taylor expansion around $x_{0}=0$. Plot (using a graphing utility) $\frac{x}{\sin x}$ and its Taylor polynomial approximations of order 6,8 and 10 , on the interval $(-10,10)$. What must you change if we were to consider the function $\frac{\cos x}{x}$ instead?
(c) Show that the function $e^{-1 / x^{2}}$ and its Taylor series expansion around $x_{0}=0$ disagree everywhere except at $x=0$. (In fact, the Taylor series around $x_{0}=0$ is identically zero, while the function $e^{-1 / x^{2}}$ is zero only at $x=0$.) Plot a figure to illustrate. Is this function analytic anywhere?
2. (Higher order equation to higher dimensional first order system) Write the following 4th order differential equation as a system of first order, linear differential equations:

$$
x^{(4)}+x^{\prime \prime}-3 x^{\prime}-\cos (t) x^{\prime}+x=t
$$

Can you state a scenario when this conversion is not possible?
3. (Nonlinear pendulum: Energy conservation) By multiplying the equation $\theta^{\prime \prime}+\frac{g}{L} \sin (\theta)=0$ by $\theta^{\prime}(t)=$ $\frac{d \theta}{d t}$, prove that the nonlinear pendulum conserves energy (the sum of the potential and kinetic energies is constant).
Deduce that the trajectories near the even critical points of the first order system are indeed closed, as the linearized systems near these points suggest.
4. (Linearization) Find the linear approximation of the function $f(x, y)=\sin \left(\pi x y^{2}\right)$ at the point $(1,1)$.
5. (Effect of higher order terms) Example 6.3.2 from Strogatz: The following example shows that the small nonlinear terms (higher order) can convert a center into a spiral (centers are delicate).
Consider the system

$$
\begin{aligned}
& x^{\prime}=-y+a x\left(x^{2}+y^{2}\right) \\
& y^{\prime}=x+a y\left(x^{2}+y^{2}\right)
\end{aligned}
$$

where $a$ is a parameter. Show that the linearized system incorrectly predicts that the origin is a center for all values of $a$, whereas in fact the origin is a stable spiral if $a<0$ and an unstable spiral if $a>0$. (Hint: for the nonlinear system, write as a first order nonlinear system in polar coordinates.)
6. (Linearized system) For the following nonlinear system, find the critical points and describe the behavior of the associated linearized system, then draw the phase portrait of the nonlinear system (make sure to check whether the linearized system accurately describe the local behavior near the equilibrium points).

$$
\begin{aligned}
x^{\prime} & =x+y^{2} \\
y^{\prime} & =2 y
\end{aligned}
$$

Check that your phase portrait is correct using a phase portrait plotting utility.

