Math 441 (Spring 2017) Assignment Four

1. (**Exponential of a matrix**) Prove that if A is similar to B then e^A is similar to e^B. How does this help with solving linear dynamical systems?

2. (**General solution- complex eigenvalues**) Consider the harmonic oscillator equation

$$x'' + 4x = 0.$$

   (a) Find the general solution using methods from 'higher order ODEs with constant coefficients'.

   (b) Rewrite the equation as a two dimensional first order system and find the complex valued general solution.

   (c) Find the real valued general solution.

   (d) Plot the phase portrait.

3. (**Two dimensional systems with radial symmetry- polar coordinates**) Consider the system

$$\begin{cases} x' = y + x(1 - x^2 - y^2) \\ y' = -x + y(1 - x^2 - y^2) \end{cases}$$

   (a) Express the system in polar coordinates (r, θ).

   (b) Use a phase portrait graphing utility to graph the phase portrait.

   (c) Prove that the origin is an unstable spiral.

   (d) By examining the phase portrait, how do the trajectories behave near the circle $r = 1$? This is called a **limit cycle**.

4. (**Lipschitz continuity**) Give an example of a function which is continuous on an interval but not Lipschitz continuous.

5. (**Lipschitz continuity**) Prove that the following functions are Lipschitz continuous on $\mathbb{R}$.

   (a) $f(x) = \sqrt{x^2 + 4}$.

   (b) $f(x) = \sin(x)$.

6. (**Existence and uniqueness counterexamples**)

   (a) Losing differentiability costs uniqueness: Consider the differential equation $x' = 3x^{2/3}$ with $x(0) = 0$. Prove that $x(t) = 0$ is a solution, and that $x(t) = t^3$ is also a solution! Prove that this IVP has infinitely many solutions (write them down)!

   (b) Discuss the existence and uniqueness of solutions of the equation $x' = x^a$ where $a > 0$ and $x(0) = 0$.

   (c) (**Local existence only: existence for only a finite interval of time**) Write the general solution of the equation $x' = 1 + x^2$, and show that it can only exist in the time interval $(-c - \pi/2, -c + \pi/2)$ where c is a constant (hence the solution cannot be extended beyond this interval). This is called *finite time blowup*.