Math 441 Analysis and Dynamics of Differential Equations
Written Assignment 6

Read your lecture notes, chapters 9, 10, 11 and 12. Submit the following problems.

Topics: Three dimensional flows: Chaos, strange attractors, fractals, Liapunov exponent.

1. Define chaos, strange attractor, fractal, Liapunov exponent.

2. Consider the Lorentz equations

$$\begin{cases} x' = \sigma(y - x) \\ y' = rx - y - xz \\ z' = xy - bz, \end{cases}$$

where σ, r, and b are constant parameters. The fixed points for this system are $(0,0,0)$ when $r \leq 1$ and $(0,0,0)$, $C^+ = (\sqrt{b(r-1)}, \sqrt{b(r-1)}, r-1)$, $C^- = (-\sqrt{b(r-1)}, -\sqrt{b(r-1)}, r-1)$ when $r > 1$.

   (a) Use linear analysis to show that when $0 < r < 1$, $(0,0,0)$ is a stable node and when $r > 1$, $(0,0,0)$ is unstable.

   (b) Using the potential function $V(x,y,z) = \frac{1}{\sigma}x^2 + y^2 + z^2$, show that the origin is globally stable when $0 < r < 1$.

   (c) Show that if $\sigma - b - 1 > 0$, then C^+ and C^- are linearly stable for $1 < r < r_H = \frac{\sigma(\sigma+b+3)}{\sigma-b-1}$ and that they are linearly unstable for $r > r_H$. (If $\sigma - b - 1 \leq 0$ then C^+ and C^- are always linearly stable.)

   (d) Show that there is a Hopf bifurcation at the critical point $r = r_H$. (This is actually a subcritical Hopf bifurcation.)

   (e) Numerically simulate the solution for $\sigma = 10$, $b = 8/3$, and $r = 28$. Make an $x(t)$, $y(t)$ and $z(t)$ plots in terms of t, as well as an (x,z) plot and an (x,y,z) plot. Describe the strange attractor that you observe.

3. Show that for the Lorenz system, the volumes in the phase space shrink exponentially fast.

4. Compute the fractal dimension of the Sierpinski triangle.