Math 441 (Spring 2017) Midterm Exam

Name:

Answer the following to the best of your knowledge

1. Give the mathematical definitions of a dynamical system, solution of a dynamical system, and a trajectory.

2. (Linearization of a function) Find the linear approximation of the function $f(x, y) = \sin(\pi x y^2)$ at the point $(1, 1)$.

3. (Fixed point for which linearization is inconclusive) Strogatz problem 6.3.10. Consider the planar system

$$x' = xy$$
$$y' = x^2 - y$$

   (a) Show that linearization predicts that the origin is a non-isolated fixed point.
   (b) Show that the origin is in fact an isolated fixed point.
   (c) Sketch the vector field in the phase plane and use this information to sketch the phase portrait.

4. (Nonlinear terms can change a star into a spiral) Strogatz problem 6.3.11. Consider the following planar system in polar coordinates

$$r' = -r$$
$$\theta' = \frac{1}{\ln r}$$

   (a) Write down the general solution $(r(t), \theta(t))$ given initial conditions (r_0, θ_0).

   (b) Show that $r(t) \to 0$ and $|\theta(t)| \to \infty$ as $t \to \infty$. Thus the origin is a stable spiral for the nonlinear system. Plot the phase portrait.

   (c) Write the system in x, y coordinates.

   (d) Linearize the above system and show that the origin is a stable star for the linearized system (a sensitive case).

5. (Conservative systems) Strogatz problem 6.5.1. Consider the system

$$x'' = x^3 - x.$$

(a) Find all equilibrium points and classify them.

(b) Find a conserved quantity.

(c) Sketch the phase portrait using the level sets of the conserved quantity.

6. State all two by two matrix canonical forms.

7. (a) Write down the general solution of

$$\vec{x}'(t) = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \vec{x}(t).$$

(b) Plot the phase portrait for $\lambda > 0$.

(c) Explicitly write down the powers $(tA)^k$ of the matrix $tA = \begin{pmatrix} t\lambda & t \\ 0 & t\lambda \end{pmatrix}$.

(d) (Exponential of a matrix) Find the general solution of the above linear system using the exponential of a matrix. Make sure the solution you obtained in part (a) and this one match.