Math 441 Analysis and Dynamics of Differential Equations

1 Introduction

1. What is a dynamical system?
2. The dimension of a dynamical system.
3. History of dynamics.
4. Classification of phenomena according to dimension and linearity.
5. Phase portrait.

2 One dimensional flows

1. $\dot{x} = f(x)$ is one dimensional while $\dot{x} = f(x,t)$ is two dimensional.
2. **Theory**:
   - Well posed-ness for forward and inverse problems.
   - Existence and uniqueness of solutions: Picard's theorem. Picard's iteration. Proof and examples. Ideas: fixed point iteration, sequences of functions, uniform convergence, Cauchy sequence.
3. **Nonlinear systems** - interesting behavior that could happen: non-uniqueness, finite time blow-up, finite time approach to critical point, singularities and behavior near singularities.
4. **Geometric representation of solutions**: (All solutions in one dimension either settle down to equilibrium or head out to infinity! No oscillations.)
   (a) Slope field (time is explicit). Analytical solutions plots (time is explicit). Impossibility of oscillations of solutions for one dimensional flows.
   (b) Phase portrait (time is implicit): One dimensional line, with critical points. Stability: qualitative analysis.
5. **Stability of critical points**:
   (a) **Linear stability**: Linearizing near a critical point (Taylor theorem, etc.).
   (b) **Local and global stability**.
   (c) **Potentials (energies)**: $\dot{x} = -\frac{d}{dx}(V(x))$. Local and global stability using potentials or *gradient flows*.
6. **Numerical solutions**: Euler's method, Improved Euler, Runge-Kutta. Error characterization: local and global errors. Importance of analysis and a good understanding of the ODE, true solution, numerical method and its stability, convergence and error before making conclusions: Examples where numerics are misleading.

2.1 One dimensional flows- Bifurcation

The dynamical system has parameters, and the qualitative structure of the flow can change (*bifurcations*, or, splitting into two branches) as the parameters are varied. Critical values of the parameters at which change occurs are *bifurcation points*. Representation: bifurcation diagrams.

1. Saddle- node bifurcation, or fold bifurcation, or turning point bifurcation (e.g. $\dot{x} = r + x^2$, $\dot{x} = r - x^2$ are prototypical or *normal* forms).
2. Transcritical bifurcation (normal form: $\dot{x} = rx - x^2$).
   - Laser threshold.
3. Pitchfork bifurcation (normal form: super critical $\dot{x} = rx - x^3$, subcritical $rx + x^3$).
   - Over-damped mechanical system. Dimensional analysis. Regular and singular perturbations.
4. Imperfect bifurcations and catastrophes ($\dot{x} = h + rx - x^3$, h is an *imperfection parameter*).
   - Insect outbreak.

3 Two dimensional flows

1. Oscillatory solutions allowed. Orbits.
2. **Nonlinear systems**: $\vec{x}'(t) = \vec{f}(\vec{x}(t))$: critical (stationary) points, linearizing (analysis, sensitive cases: where linearization may only give partial information about the dynamics near the stationary point, Jacobian), phase portraits from linearization, limit cycles in nonlinear flows.
3. **Linear systems**: $\vec{x}'(t) = A\vec{x}(t)$: analytical solutions (real and complex eigenvalues, repeated eigenvalues including defective case), types of critical points, all possible phase portraits.

4. **Computer graphing**: phase portrait generators. Graphing analytical solutions in terms of t. Numerically solving the ODE (again solution in terms of t).

5. **Limit cycles** in nonlinear flows:

   (a) What is a limit cycle?

   (b) Limit cycle stability types.

   (c) How do we know a system has a limit cycle? Steps in that direction:

   - Poincare Bendixon theorem. Other than proving that a limit cycle exists, this is a key theorem in nonlinear dynamics since it implies that chaos cannot occur in the phase plane.

   (d) How do we know a system cannot have a limit cycle?

   - Gradient systems cannot have limit cycles.
   - Systems with Liaponuv functions cannot have limit cycles.
   - Dulac's criterion (based on Green's theorem).
   - Index theory.
   - others?

   (e) How many limit cycles does a given system have?

   (f) If a system has a limit cycle, what's the shape of the limit cycle? Stability? Period?

   (g) Examples:

   - $$\begin{cases} x' = -y + x(1 - x^2 - y^2) \\ y' = x + y(1 - x^2 - y^2) \end{cases}$$
   - Leinard systems.
   - Vander Pol equation.

6. α **and** ω **limit sets**, their properties, and proof of Poincare Bendixon theorem.

7. **Hopf bifurcation** (subcritical and super critical) and examples.

4 Three Dimensional Flows and Chaos

1. How did it all start? Laplace's demon. Lorentz weather model (1970s). The butterfly effect.

2. Chaos (indeterminism at its best).

3. Famous examples:

   - Lorentz equations (simplified few Navier Stokes equations modeling fluid dynamics):
   $$\begin{cases} \dot{x} = P(y - x) \\ \dot{y} = Rx - y - xz \\ \dot{z} = xy - By \end{cases}$$
   - Rossler system (arose from studying oscillations in chemical reactions)
   $$\begin{cases} \dot{x} = -(y + z) \\ \dot{y} = x + Ay \\ \dot{z} = B + xz - Cz \end{cases}$$

4. Chaos on strange attractors.

   - Henon's attractor,
   $$\begin{cases} x_{n+1} = (y_n + 1) - (1.4x_n^2) \\ y_{n+1} = 0.3x_n \\ \quad x_0 = 1, y_0 = 1 \end{cases}$$

5. Fractals:

   - Dimension of self- similar fractals $D = \log(N)/\log(1/r)$ (for a self similar object of N parts scaled down by a factor r).
   - Iterated Function Systems (IFS)
   $$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix}$$

6. Liapunov exponent.

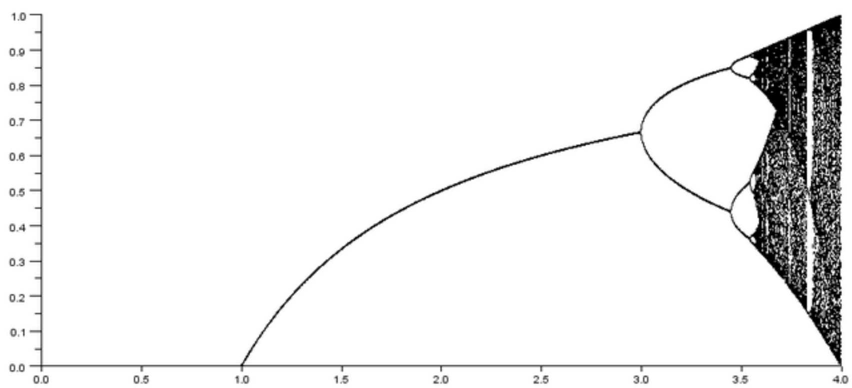

Figure 1: A bifurcation diagram (with chaos)