Math 441 Analysis and Dynamics of Differential Equations

1 First half of the semester

1. What is a dynamical system? (Mathematical definition: differential equations approach). Difference between a differential equation and a dynamical system.
2. What is the dimension of a dynamical system?
3. What is chaos?
4. Why is it easier to study linear systems? (Principle of superposition of solutions.)
5. Why is it important to study nonlinear systems?
6. History of dynamics.
7. Classification of phenomena according to dimension and linearity (overview).
8. Phase portrait (axes: state variables, time: implicit, critical points).
9. Describing the solution qualitatively (without obtaining an analytical solution): critical points (stationary), stability analysis, phase portrait, slope field, analysis of differential equations techniques (boundedness, regularity, *etc.*).
10. Basic examples:
    - Natural and logistic population growth models; bifurcation.
    - The linear and nonlinear pendulum:
      Nonlinear pendulum: modeling, transforming higher order differential equations into a first order system of equations in higher dimensions, critical points, stability, phase portrait and interpretation, energy conservation.
      Linear pendulum: small angle approximation, linearization, analytical solution, phase portrait and interpretation.
11. Taylor expansion and Taylor theorem. Difference between smooth and analytic real functions. (No such difference for differentiable complex functions).
12. Linearization around a critical point. Taylor's theorem. Higher order terms. First derivative significance and the Jacobian. What if those are zero at the critical point? When does the behavior of the linearized system transcend to the nonlinear system (theorems)?
13. Examples we've worked out in class:
    (a) $x'(t) = ax$.
    (b) $\vec{x}'(t) = A\vec{x}$.
    (c) Logistic equation $x'(t) = kx(M-x)$. Logistic equation with harvesting $x'(t) = kx(M-x)-h$ (and bifurcation).
    (d) Nonlinear pendulum $\theta'' + \frac{g}{L}\sin(\theta) = 0$.
    (e) Linear pendulum $\theta'' + \frac{g}{L}\theta = 0$.
    (f) Examples of first order ODEs and their phase portraits.
    (g) Examples of linear and nonlinear systems in two dimensions.
14. Types of critical points in the phase plane (a topic related to two dimensional dynamical systems): derivation, repeated eigenvalues, canonical forms, sensitive cases.
15. Conservative systems, reversible systems. Nonlinear centers in conservative and reversible systems.
16. Planar systems with inherent radial symmetry: analysis using polar coordinates.
17. Matrix canonical forms. Similarity in matrices. Exponential of a matrix (definition, properties, solving linear dynamical systems).

2 Second half of the semester

1. Existence and Uniqueness theorem and proof. Picard's iteration.
2. Well-posedness. Continuous dependence on data. Gronwall's inequality.
3. Two dimensional systems: Poicare Bendixon Theorem, examples and proof.
4. Three dimensional systems and chaos.

3 Summary- According to Dimension

3.1 One dimensional flows

1. $\dot{x} = f(x)$ is one dimensional while $\dot{x} = f(x,t)$ is two dimensional.
2. **Theory**:
   - Well posed-ness for forward and inverse problems.
   - Existence and uniqueness of solutions: Picard's theorem. Picard's iteration. Proof and examples. Ideas: fixed point iteration, sequences of functions, uniform convergence, Cauchy sequence.
3. **Nonlinear systems** - interesting behavior that could happen: non-uniqueness, finite time blow-up, finite time approach to critical point, singularities and behavior near singularities.
4. **Geometric representation of solutions**: (All solutions in one dimension either settle down to equilibrium or head out to infinity! No oscillations.)
   (a) Slope field (time is explicit). Analytical solutions plots (time is explicit). Impossibility of oscillations of solutions for one dimensional flows.
   (b) Phase portrait (time is implicit): One dimensional line, with critical points. Stability: qualitative analysis.
5. **Stability of critical points**:
   (a) **Linear stability**: Linearizing near a critical point (Taylor theorem, etc.).
   (b) **Local and global stability**.
   (c) **Potentials (energies)**: $\dot{x} = -\frac{d}{dx}(V(x))$. Local and global stability using potentials or *gradient flows*.
6. **Numerical solutions**: Euler's method, Improved Euler, Runge-Kutta. Error characterization: local and global errors. Importance of analysis and a good understanding of the ODE, true solution, numerical method and its stability, convergence and error before making conclusions: Examples where numerics are misleading.

3.2 Two dimensional flows

1. Oscillatory solutions allowed. Orbits.
2. **Nonlinear systems**: $\vec{x}'(t) = \vec{f}(\vec{x}(t))$: critical (stationary) points, linearizing (analysis, sensitive cases: where linearization may only give partial information about the dynamics near the stationary point, Jacobian), phase portraits from linearization, limit cycles in nonlinear flows.
3. **Linear systems**: $\vec{x}'(t) = A\vec{x}(t)$: analytical solutions (real and complex eigenvalues, repeated eigenvalues including defective case), types of critical points, all possible phase portraits.
4. **Computer graphing**: phase portrait generators. Graphing analytical solutions in terms of t. Numerically solving the ODE (again solution in terms of t).
5. **Limit cycles** in nonlinear flows:
   (a) What is a limit cycle?
   (b) Limit cycle stability types.
   (c) How do we know a system has a limit cycle? Steps in that direction:
      - Poincare Bendixon theorem. Other than proving that a limit cycle exists, this is a key theorem in nonlinear dynamics since it implies that chaos cannot occur in the phase plane.
   (d) How do we know a system cannot have a limit cycle?
      - Gradient systems cannot have limit cycles.
      - Systems with Liaponuv functions cannot have limit cycles.
      - Dulac's criterion (based on Green's theorem).
      - Index theory.
      - others?
   (e) How many limit cycles does a given system have?
   (f) If a system has a limit cycle, what's the shape of the limit cycle? Stability? Period?
   (g) Examples:
      - $$\begin{cases} x' = -y + x(1 - x^2 - y^2) \\ y' = x + y(1 - x^2 - y^2) \end{cases}$$
      - Leinard systems.
      - Vander Pol equation.
6. α **and** ω **limit sets**, their properties, and proof of Poincare Bendixon theorem.
7. **Hopf bifurcation** (subcritical and super critical) and examples.

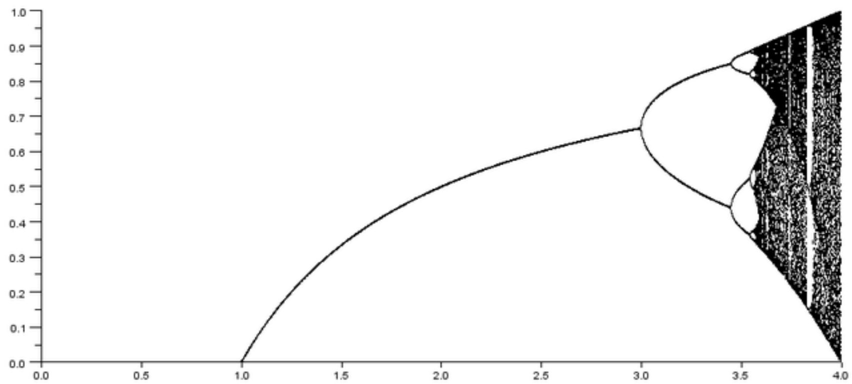

Figure 1: A bifurcation diagram (with chaos)

3.3 Three Dimensional Flows and Chaos

1. How did it all start? Laplace's demon. Lorentz weather model (1970s). The butterfly effect.

2. Chaos (indeterminism at its best).

3. Famous examples:

   - Lorentz equations (simplified few Navier Stokes equations modeling fluid dynamics):

$$\begin{cases} \dot{x} = P(y-x) \\ \dot{y} = Rx - y - xz \\ \dot{z} = xy - By \end{cases}$$

   - Rossler system (arose from studying oscillations in chemical reactions)

$$\begin{cases} \dot{x} = -(y+z) \\ \dot{y} = x + Ay \\ \dot{z} = B + xz - Cz \end{cases}$$

4. Chaos on strange attractors.

   - Henon's attractor,

$$\begin{cases} x_{n+1} = (y_n + 1) - (1.4x_n^2) \\ y_{n+1} = 0.3x_n \\ \quad x_0 = 1, y_0 = 1 \end{cases}$$

5. Fractals:

   - Dimension of self- similar fractals $D = \log(N)/\log(1/r)$ (for a self similar object of N parts scaled down by a factor r).
   - Iterated Function Systems (IFS)

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix}$$

6. Liapunov exponent.