Math 441 Midterm Exam Spring 2016
Name:

Answer the following to the best of your knowledge.

1. Using linear stability analysis, classify the fixed points of the *Gompertz model of tumor growth*

$$N'(t) = -aN(t)\ln(bN(t)),$$

where $N(t)$ is proportional to the number of cells in the tumor, and $a > 0$, $b > 0$ are parameters.

2. In statistical mechanics, the phenomenon of *critical slowing down* is a signature of a second order phase transition. At the transition, the system relaxes to equilibrium much more slowly than usual. Mathematically, the ODE

$$x'(t) = -x^3(t)$$

reflects the effect.

   (a) Find the analytical solution of the ODE with an arbitrary initial condition x_0.

   (b) Show that $x(t) \to 0$ as $t \to \infty$, but that the decay is not exponential.

   (c) Make a numerically accurate plot of the solution for the initial condition $x_0 = 10$, $0 \leq t \leq 10$. Then, on the same graph, plot the solution of $x'(t) = -x(t)$ for the same initial condition.

3. (a) Give an example of an ODE where the solution blows up in finite time.

   (b) Can this happen for linear ODEs? Why or why not?

   (c) Does this violate the existence and uniqueness theorem for first order ODEs $x'(t) = f(t, x)$? Explain.

4. Plot the potential function $V(x)$ of $x'(t) = 2 + \sin(x)$ and identify all the equilibrium points and there stability. What type of stability are you analyzing?

5. (a) Find the values of r at which bifurcations for the ODE $x'(t) = 5 - re^{-x^2}$ occur, and classify those as saddle-node, transcritical, supercritical pitchfork, or subcritical pitchfork, then sketch a bifurcation diagram of the critical points x^* vs. r.

   (b) Use Taylor expansions to write the RHS of the ODE in its normal form.

6. Explain the difference between regular perturbation vs. singular perturbation using the ODEs $\epsilon x'(t) + x(t) = 0$, $x(0) = 1$, and $x'(t) + \epsilon x(t) = 0$, $x(0) = 1$.

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