It seems from these two constructions that bisecting a line segment and bisecting an angle are virtually the same problem. Euclid bisects the angle before the line segment, but he uses two similar constructions (*Elements*, Propositions 9 and 10 of Book I). However, a distinction between line segments and angles emerges when we attempt division into three or more parts. There is a simple tool for dividing a line segment in any number of equal parts—*parallel lines*—but no corresponding tool for dividing angles.

Constructing the parallel to a line through a given point

We use the two constructions of perpendiculars noted above—for a point off the line and a point on the line. Given a line $\mathscr{L}$ and a point P outside $\mathscr{L}$, first construct the perpendicular line $\mathscr{M}$ to $\mathscr{L}$ through P. Then construct the perpendicular to $\mathscr{M}$ through P, which is the parallel to $\mathscr{L}$ through P.

Dividing a line segment into n equal parts

Given a line segment AB, draw any other line $\mathscr{L}$ through A and mark n successive, equally spaced points $A_1, A_2, A_3, \ldots, A_n$ along $\mathscr{L}$ using the compass set to any fixed radius. Figure 1.8 shows the case $n = 5$. Then connect A_n to B, and draw the parallels to BA_n through $A_1, A_2, \ldots, A_{n-1}$. These parallels divide AB into n equal parts.

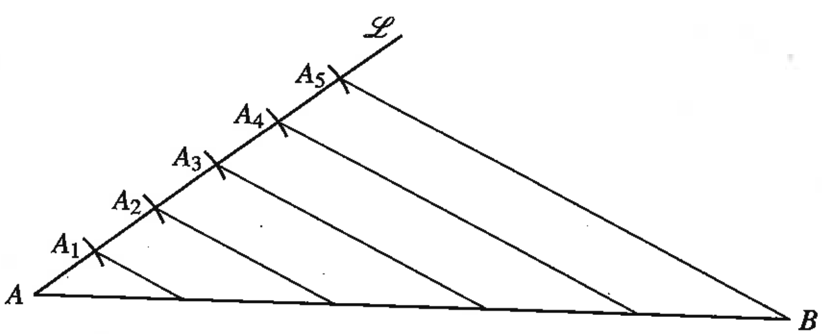

Figure 1.8: Dividing a line segment into equal parts

This construction depends on a property of parallel lines sometimes attributed to Thales (Greek mathematician from around 600 BCE): *parallels cut any lines they cross in proportional segments*. The most commonly used instance of this theorem is shown in Figure 1.9, where a parallel to one side of a triangle cuts the other two sides proportionally.

The line $\mathscr{L}$ parallel to the side BC cuts side AB into the segments AP and PB, side AC into AQ and QC, and $|AP|/|PB| = |AQ|/|QC|$.

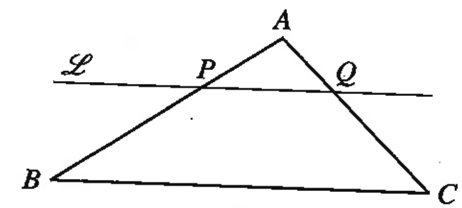

Figure 1.9: The Thales theorem in a triangle

This theorem of Thales is the key to using algebra in geometry. In the next section we see how it may be used to multiply and divide line segments, and in Chapter 2 we investigate how it may be derived from fundamental geometric principles.

Exercises

1.3.1 Check for yourself the constructions of perpendiculars and parallels described in words above.

1.3.2 Can you find a more direct construction of parallels?

Perpendiculars give another important polygon—the square.

1.3.3 Give a construction of the square on a given line segment.

1.3.4 Give a construction of the square tiling of the plane.

One might try to use division of a line segment into n equal parts to divide an angle into n equal parts as shown in Figure 1.10. We mark A on OP and B at equal distance on OQ as before, and then try to divide angle POQ by dividing line segment AB. However, this method is faulty even for division into three parts.

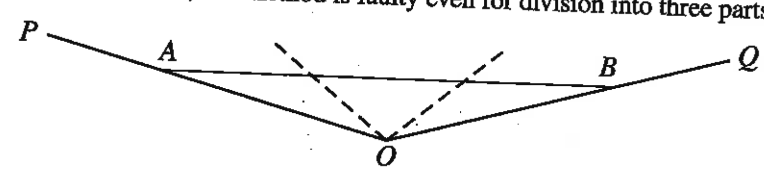

Figure 1.10: Faulty trisection of an angle

1.3.5 Explain why division of AB into three equal parts (trisection) does *not* always divide angle POQ into three equal parts. (Hint: Consider the case in which POQ is nearly a straight line.)

The version of the Thales theorem given above (referring to Figure 1.9) has an equivalent form that is often useful.

1.3.6 If A, B, C, P, Q are as in Figure 1.9, so that $|AP|/|PB| = |AQ|/|QC|$, show that this equation is equivalent to $|AP|/|AB| = |AQ|/|AC|$.

1.4 Multiplication and division

Not only can one add and subtract line segments (Section 1.1); one can also multiply and divide them. The *product* ab and *quotient* a/b of line segments a and b are obtained by the straightedge and compass constructions below. The key ingredients are parallels, and the key geometric property involved is the Thales theorem on the proportionality of line segments cut off by parallel lines.

To get started, it is necessary to choose a line segment as the *unit of length*, 1, which has the property that $1a = a$ for any length a.

Product of line segments

To multiply line segment b by line segment a, we first construct any triangle UOA with $|OU| = 1$ and $|OA| = a$. We then extend OU by length b to B_1 and construct the parallel to UA through B_1. Suppose this parallel meets the extension of OA at C (Figure 1.11).

By the Thales theorem, $|AC| = ab$.

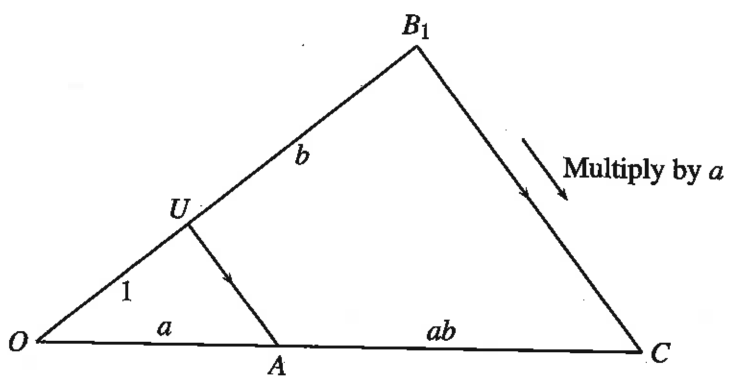

Figure 1.11: The product of line segments

Quotient of line segments

To divide line segment b by line segment a, we begin with the same triangle UOA with $|OU| = 1$ and $|OA| = a$. Then we extend OA by distance b to B_2 and construct the parallel to UA through B_2. Suppose that this parallel meets the extension of OU at D (Figure 1.12).

By the Thales theorem, $|UD| = b/a$.

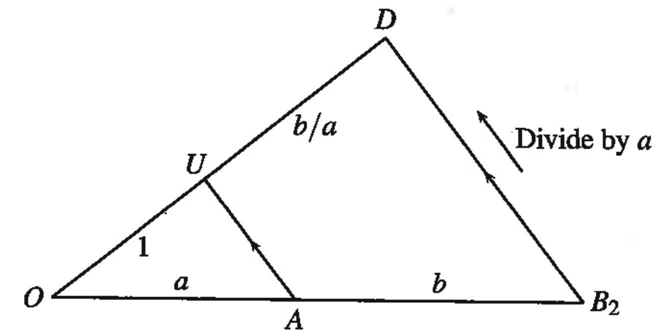

Figure 1.12: The quotient of line segments

The sum operation from Section 1.1 allows us to construct a segment n units in length, for any natural number n, simply by adding the segment 1 to itself n times. The quotient operation then allows us to construct a segment of length m/n, for any natural numbers m and $n \neq 0$. These are what we call the *rational* lengths. A great discovery of the Pythagoreans was that *some lengths are not rational*, and that some of these "irrational" lengths can be constructed by straightedge and compass. It is not known how the Pythagoreans made this discovery, but it has a connection with the Thales theorem, as we will see in the next section.

Exercises

Exercise 1.3.6 showed that if PQ is parallel to BC in Figure 1.9, then $|AP|/|AB| = |AQ|/|AC|$. That is, a parallel implies proportional (left and right) sides. The following exercise shows the converse: proportional sides imply a parallel, or (equivalently), a nonparallel implies nonproportional sides.

1.4.1 Using Figure 1.13, or otherwise, show that if PR is *not* parallel to BC, then $|AP|/|AB| \neq |AR|/|AC|$.

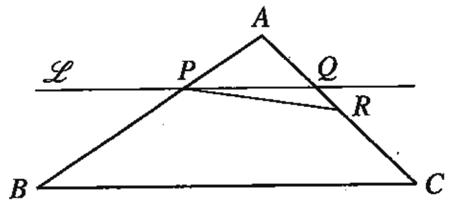

Figure 1.13: Converse of the Thales theorem

1.4.2 Conclude from Exercise 1.4.1 that if P is any point on AB and Q is any point on AC, then PQ is parallel to BC *if and only if* $|AP|/|AB| = |AQ|/|AC|$.

The "only if" direction of Exercise 1.4.2 leads to two famous theorems—the *Pappus* and *Desargues theorems*—that play an important role in the foundations of geometry. We will meet them in more general form later. In their simplest form, they are the following theorems about parallels.

1.4.3 (Pappus of Alexandria, around 300 CE) Suppose that A,B,C,D,E,F lie alternately on lines $\mathscr{L}$ and $\mathscr{M}$ as shown in Figure 1.14.

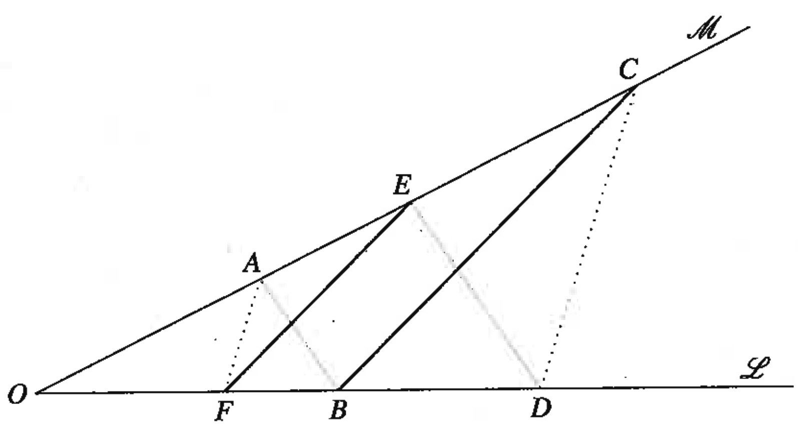

Figure 1.14: The parallel Pappus configuration

Use the Thales theorem to show that if AB is parallel to ED and FE is parallel to BC then

$$\frac{|OA|}{|OF|} = \frac{|OC|}{|OD|}.$$

Deduce from Exercise 1.4.2 that AF is parallel to CD.

1.4.4 (Girard Desargues, 1648) Suppose that points A,B,C,A',B',C' lie on concurrent lines $\mathscr{L},\mathscr{M},\mathscr{N}$ as shown in Figure 1.15. (The triangles ABC and $A'B'C'$ are said to be "in perspective from O.")

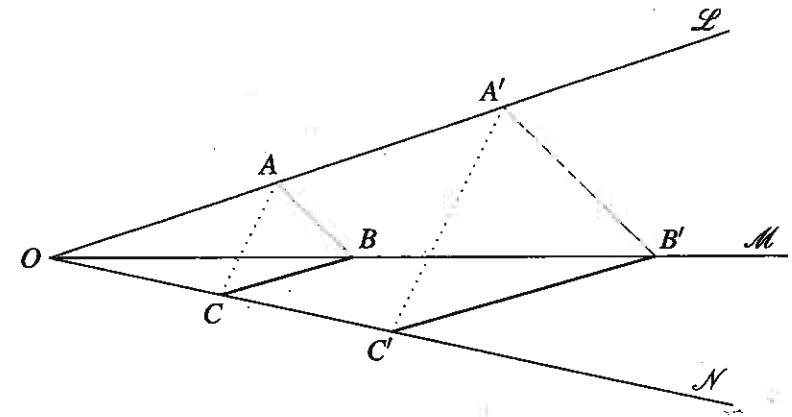

Figure 1.15: The parallel Desargues configuration

Use the Thales theorem to show that if AB is parallel to $A'B'$ and BC is parallel to $B'C'$, then

$$\frac{|OA|}{|OC|} = \frac{|OA'|}{|OC'|}.$$

Deduce from Exercise 1.4.2 that AC is parallel to $A'C'$.

1.5 Similar triangles

Triangles ABC and $A'B'C'$ are called *similar* if their corresponding angles are equal, that is, if

$$\begin{aligned}
\text{angle at } A &= \text{angle at } A' \quad (= \alpha \text{ say}),\\
\text{angle at } B &= \text{angle at } B' \quad (= \beta \text{ say}),\\
\text{angle at } C &= \text{angle at } C' \quad (= \gamma \text{ say}).
\end{aligned}$$

It turns out that equal angles imply that *all sides are proportional*, so we may say that one triangle is a magnification of the other, or that they have the same "shape." This important result extends the Thales theorem, and actually follows from it.

Why similar triangles have proportional sides

Imagine moving triangle ABC so that vertex A coincides with A' and sides AB and AC lie on sides $A'B'$ and $A'C'$, respectively. Then we obtain the situation shown in Figure 1.16. In this figure, b and c denote the side lengths of triangle ABC opposite vertices B and C, respectively, and b' and c' denote the side lengths of triangle $A'B'C'(=AB'C')$ opposite vertices B' and C', respectively.

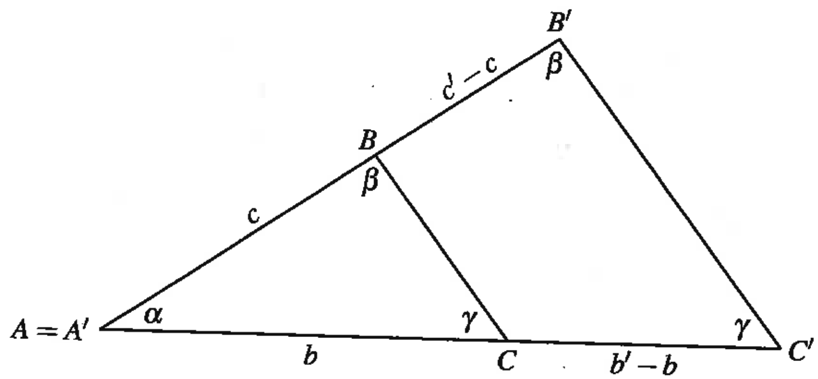

Figure 1.16: Similar triangles

Because BC and $B'C'$ both meet AB' at angle β, they are parallel, and so it follows from the Thales theorem (Section 1.3) that

$$\frac{b}{c}=\frac{b'-b}{c'-c}.$$

Multiplying both sides by $c(c'-c)$ gives $b(c'-c)=c(b'-b)$, that is,

$$bc'-bc=cb'-cb,$$

and hence

$$bc'=cb'.$$

Finally, dividing both sides by cc', we get

$$\frac{b}{c}=\frac{b'}{c'}.$$

That is, *corresponding sides of triangles ABC and A'B'C' opposite to the angles β and γ are proportional.*

We got this result by making the angles α in the two triangles coincide. If we make the angles β coincide instead, we similarly find that the sides opposite to α and γ are proportional. Thus, in fact, *all corresponding sides of similar triangles are proportional.* □

This consequence of the Thales theorem has many implications. In everyday life, it underlies the existence of scale maps, house plans, engineering drawings, and so on. In pure geometry, its implications are even more varied. Here is just one, which shows why square roots and irrational numbers turn up in geometry.

The diagonal of the unit square is $\sqrt{2}$

The diagonals of the unit square cut it into four quarters, each of which is a triangle similar to the half square cut off by a diagonal (Figure 1.17):

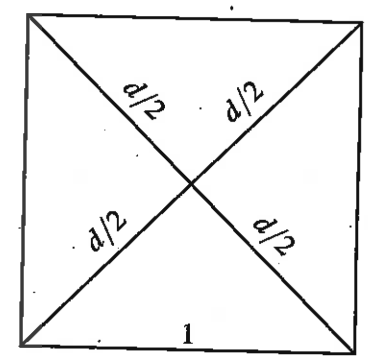

Figure 1.17: Quarters and halves of the square

Each of the triangles in question has one right angle and two half right angles, so it follows from the theorem above that corresponding sides of any two of these triangles are proportional. In particular, if we take the half square, with short side 1 and long side d, and compare it with the quarter square, with short side $d/2$ and long side 1, we get

$$\frac{\text{short}}{\text{long}}=\frac{1}{d}=\frac{d/2}{1}.$$

Multiplying both sides of the equation by $2d$ gives $2=d^2$, so $d=\sqrt{2}$. □

The great, but disturbing, discovery of the Pythagoreans is that $\sqrt{2}$ *is irrational*. That is, there are no natural numbers m and n such $\sqrt{2} = m/n$.

If there are such m and n we can assume that they have no common divisor, and then the assumption $\sqrt{2} = m/n$ implies

	$2 = m^2/n^2$	squaring both sides
hence	$m^2 = 2n^2$	multiplying both sides by n^2
hence	m^2 is even	
hence	m is even	since the square of an odd number is odd
hence	$m = 2l$	for some natural number l
hence	$m^2 = 4l^2 = 2n^2$	
hence	$n^2 = 2l^2$	
hence	n^2 is even	
hence	n is even	since the square of an odd number is odd.

Thus, m and n have the common divisor 2, contrary to assumption. Our original assumption is therefore false, so there are no natural numbers m and n such that $\sqrt{2} = m/n$. □

Lengths, products, and area

Geometry obviously has to include the diagonal of the unit square, hence *geometry includes the study of irrational lengths.* This discovery troubled the ancient Greeks, because they did not believe that irrational lengths could be treated like numbers. In particular, the idea of interpreting the product of line segments as another line segment is *not* in Euclid. It first appears in Descartes' *Géométrie* of 1637, where algebra is used systematically in geometry for the first time.

The Greeks viewed the product of line segments a and b as the *rectangle* with perpendicular sides a and b. If lengths are not necessarily numbers, then the product of two lengths is best interpreted as an area, and the product of three lengths as a volume—but then the product of four lengths seems to have no meaning at all. This difficulty perhaps explains why algebra appeared comparatively late in the development of geometry. On the other hand, interpreting the product of lengths as an area gives some remarkable insights, as we will see in Chapter 2. So it is also possible that algebra had to wait until the Greek concept of product had exhausted its usefulness.

Exercises

In general, two geometric figures are called similar if one is a magnification of the other. Thus, two rectangles are similar if the ratio $\frac{\text{long side}}{\text{short side}}$ is the same for both.

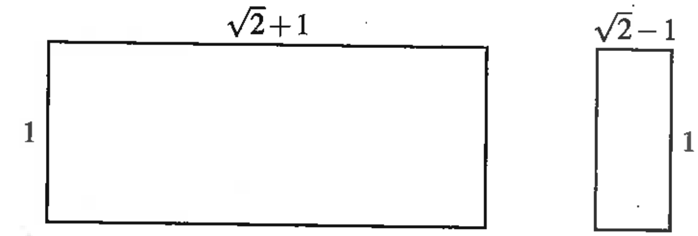

Figure 1.18: A pair of similar rectangles

1.5.1 Show that $\frac{\sqrt{2}+1}{1} = \frac{1}{\sqrt{2}-1}$ and hence that the two rectangles in Figure 1.18 are similar.

1.5.2 Deduce that if a rectangle with long side a and short side b has the same shape as the two above, then so has the rectangle with long side b and short side $a - 2b$.

This simple observation gives another proof that $\sqrt{2}$ is irrational:

1.5.3 Suppose that $\sqrt{2} + 1 = m/n$, where m and n are natural numbers with m as small as possible. Deduce from Exercise 1.5.2 that we also have $\sqrt{2} + 1 = n/(m - 2n)$. This is a contradiction. Why?

1.5.4 It follows from Exercise 1.5.3 that $\sqrt{2}+1$ is irrational. Why does this imply that $\sqrt{2}$ is irrational?

1.6 Discussion

Euclid's *Elements* is the most influential book in the history of mathematics, and anyone interested in geometry should own a copy. It is not easy reading, but you will find yourself returning to it year after year and noticing something new. The standard edition in English is Heath's translation, which is now available as a Dover reprint of the 1925 Cambridge University Press edition. This reprint is carried by many bookstores; I have even seen it for sale at Los Angeles airport! Its main drawback is its size—three bulky volumes—due to the fact that more than half the content consists of

We can certainly extend a given rectangle to a square and hence reconstruct the square on the hypotenuse. The main problem is to reconstruct the right-angled triangle, from the hypotenuse, so that the other vertex lies on the dashed line. See whether you can think of a way to do this; a really elegant solution is given in Section 2.7. Once we have the right-angled triangle, we can certainly construct the squares on its other two sides—in particular, the gray square equal in area to the gray rectangle.

Exercises

It follows from the Pythagorean theorem that a right-angled triangle with sides 3 and 4 has hypotenuse $\sqrt{3^2+4^2}=\sqrt{25}=5$. But there is *only one* triangle with sides 3, 4, and 5 (by the SSS criterion mentioned in Exercise 2.2.2), so putting together lengths 3, 4, and 5 always makes a right-angled triangle. This triangle is known as the $(3,4,5)$ triangle.

2.5.1 Verify that the $(5,12,13)$, $(8,15,17)$, and $(7,24,25)$ triangles are right-angled.

2.5.2 Prove the converse Pythagorean theorem: If $a,b,c>0$ and $a^2+b^2=c^2$, then the triangle with sides a,b,c is right-angled.

2.5.3 How can we be sure that lengths $a,b,c>0$ with $a^2+b^2=c^2$ actually fit together to make a triangle? (*Hint:* Show that $a+b>c$.)

Right-angled triangles can be used to construct certain irrational lengths. For example, we saw in Section 1.5 that the right-angled triangle with sides 1, 1 has hypotenuse $\sqrt{2}$.

2.5.4 Starting from the triangle with sides 1, 1, and $\sqrt{2}$, find a straightedge and compass construction of $\sqrt{3}$.

2.5.5 Hence, obtain constructions of $\sqrt{n}$ for $n=2,3,4,5,6,\ldots$.

2.6 Proof of the Thales theorem

We mentioned this theorem in Chapter 1 as a fact with many interesting consequences, such as the proportionality of similar triangles. We are now in a position to prove the theorem as Euclid did in his Proposition 2 of Book VI. Here again is a statement of the theorem.

The Thales theorem. *A line drawn parallel to one side of a triangle cuts the other two sides proportionally.*

The proof begins by considering triangle ABC, with its sides AB and AC cut by the parallel PQ to side BC (Figure 2.15). Because PQ is parallel to BC, the triangles PQB and PQC on base PQ have the same height, namely the distance between the parallels. They therefore have the same area.

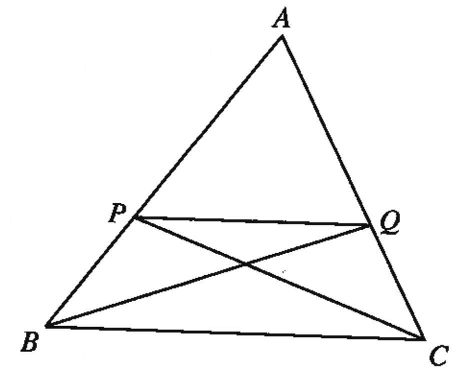

Figure 2.15: Triangle sides cut by a parallel

If we add triangle APQ to each of the equal-area triangles PQB and PQC, we get the triangles AQB and APC, respectively. Hence, the latter triangles are also equal in area.

Now consider the two triangles—APQ and PQB—that make up triangle AQB as triangles with bases on the line AB. They have the same height relative to this base (namely, the perpendicular distance of Q from AB). Hence, their bases are in the ratio of their areas:

$$\frac{|AP|}{|PB|}=\frac{\text{area } APQ}{\text{area } PQB}.$$

Similarly, considering the triangles APQ and PQC that make up the triangle APC, we find that

$$\frac{|AQ|}{|QC|}=\frac{\text{area } APQ}{\text{area } PQC}.$$

Because area PQB equals area PQC, the right sides of these two equations are equal, and so are their left sides. That is,

$$\frac{|AP|}{|PB|}=\frac{|AQ|}{|QC|}.$$

In other words, the line PQ cuts the sides AB and AC proportionally. □

Exercises

As seen in Exercise 1.3.6, $|AP|/|PB| = |AQ|/|QC|$ is equivalent to $|AP|/|AB| = |AQ|/|AC|$. This equation is a more convenient formulation of the Thales theorem if you want to prove the following generalization:

2.6.1 Suppose that there are several parallels $P_1Q_1, P_2Q_2, P_3Q_3 \ldots$ to the side BC of triangle ABC. Show that

$$\frac{|AP_1|}{|AQ_1|} = \frac{|AP_2|}{|AQ_2|} = \frac{|AP_3|}{|AQ_3|} = \cdots = \frac{|AB|}{|AC|}.$$

We can also drop the assumption that the parallels $P_1Q_1, P_2Q_2, P_3Q_3 \ldots$ fall across a triangle ABC.

2.6.2 If parallels $P_1Q_1, P_2Q_2, P_3Q_3 \ldots$ fall across a pair of *parallel* lines $\mathscr{L}$ and $\mathscr{M}$, what can we say about the lengths they cut from $\mathscr{L}$ and $\mathscr{M}$?

2.7 Angles in a circle

The isosceles triangle theorem of Section 2.2, simple though it is, has a remarkable consequence.

Invariance of angles in a circle. *If A and B are two points on a circle, then, for all points C on one of the arcs connecting them, the angle ACB is constant.*

To prove invariance we draw lines from A, B, C to the center of the circle, O, along with the lines making the angle ACB (Figure 2.16).

Because all radii of the circle are equal, $|OA| = |OC|$. Thus triangle AOC is isosceles, and the angles α in it are equal by the isosceles triangle theorem. The angles β in triangle BOC are equal for the same reason.

Because the angle sum of any triangle is π (Section 2.1), it follows that the angle at O in triangle AOC is $\pi - 2\alpha$ and the angle at O in triangle BOC is $\pi - 2\beta$. It follows that the third angle at O, angle AOB, is $2(\alpha+\beta)$, because the total angle around any point is 2π. But angle AOB is *constant*, so $\alpha+\beta$ is also constant, and $\alpha+\beta$ is precisely the angle at C. □

An important special case of this theorem is when A, O, and B lie in a straight line, so $2(\alpha+\beta) = \pi$. In this case, $\alpha+\beta = \pi/2$, and thus we have the following theorem (which is also attributed to Thales).

Angle in a semicircle theorem. *If A and B are the ends of a diameter of a circle, and C is any other point on the circle, then angle ACB is a right angle.* □

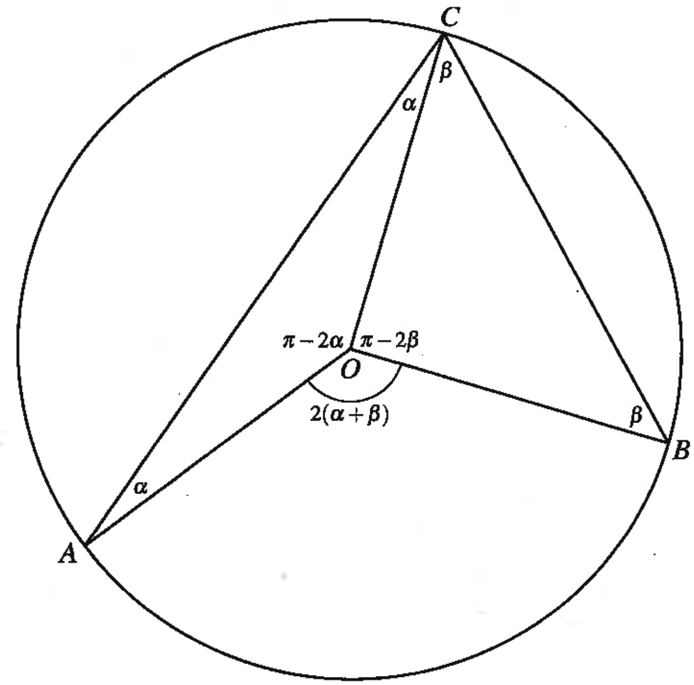

Figure 2.16: Angle $\alpha+\beta$ in a circle

This theorem enables us to solve the problem left open at the end of Section 2.5: Given a hypotenuse AB, how do we construct the right-angled triangle whose other vertex C lies on a given line? Figure 2.17 shows how.

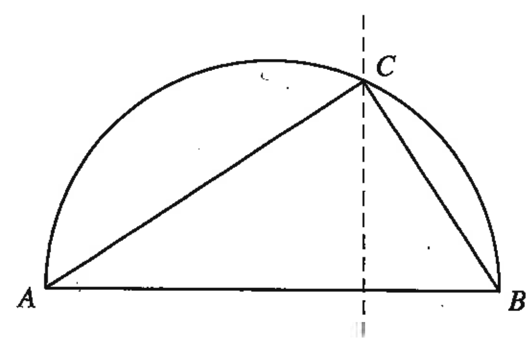

Figure 2.17: Constructing a right-angled triangle with given hypotenuse

The trick is to draw the semicircle on diameter AB, which can be done by first bisecting AB to obtain the center of the circle. Then the point where the semicircle meets the given line (shown dashed) is necessarily the other vertex C, because the angle at C is a right angle.

This construction completes the solution of the problem raised at the end of Section 2.5: finding a square equal in area to a given rectangle. In Section 2.8 we will show that Figure 2.17 also enables us to construct the *square root* of an arbitrary length, and it gives a new proof of the Pythagorean theorem.

Exercises

2.7.1 Explain how the angle in a semicircle theorem enables us to construct a right-angled triangle with a given hypotenuse AB.

2.7.2 Then, by looking at Figure 2.13 from the bottom up, find a way to construct a square equal in area to a given rectangle.

2.7.3 Given any two squares, we can construct a square that equals (in area) the sum of the two given squares. Why?

2.7.4 Deduce from the previous exercises that any polygon may be "squared"; that is, there is a straightedge and compass construction of a square equal in area to the given polygon. (You may assume that the given polygon can be cut into triangles.)

The possibility of "squaring" any polygon was apparently known to Greek mathematicians, and this may be what tempted them to try "squaring the circle": constructing a square equal in area to a given circle. There is no straightedge and compass solution of the latter problem, but this was not known until 1882.

Coming back to angles in the circle, here is another theorem about invariance of angles:

2.7.5 If a quadrilateral has its vertices on a circle, show that its opposite angles sum to π.

2.8 The Pythagorean theorem revisited

In Book VI, Proposition 31 of the *Elements*, Euclid proves a generalization of the Pythagorean theorem. From it, we get a new proof of the ordinary Pythagorean theorem, based on the proportionality of similar triangles.

Given a right-angled triangle with sides a, b, and hypotenuse c, we divide it into two smaller right-angled triangles by the perpendicular to the hypotenuse through the opposite vertex (the dashed line in Figure 2.18).

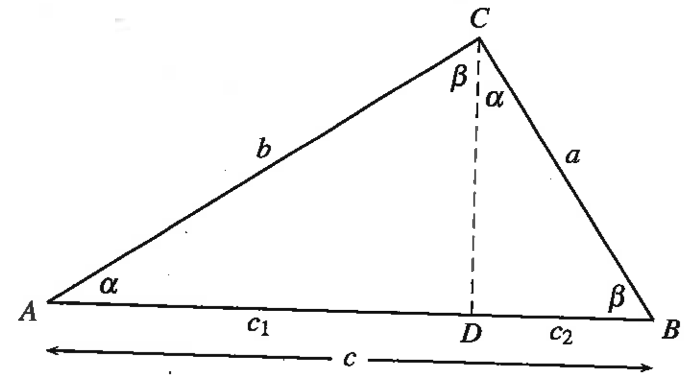

Figure 2.18: Subdividing a right-angled triangle into similar triangles

All three triangles are similar because they have the same angles α and β. If we look first at the angle α at A and the angle β at B, then

$$\alpha + \beta = \frac{\pi}{2}$$

because the angle sum of triangle ABC is π and the angle at C is $\pi/2$. But then it follows that angle $ACD = \beta$ in triangle ACD (to make its angle sum $= \pi$) and angle $DCB = \alpha$ in triangle DCB (to make its angle sum $= \pi$).

Now we use the proportionality of these triangles, calling the side opposite α in each triangle "short" and the side opposite β "long" for convenience. Comparing triangle ABC with triangle ADC, we get

$$\frac{\text{long side}}{\text{hypotenuse}} = \frac{b}{c} = \frac{c_1}{b}, \quad \text{hence} \quad b^2 = cc_1.$$

Comparing triangle ABC with triangle DCB, we get

$$\frac{\text{short side}}{\text{hypotenuse}} = \frac{a}{c} = \frac{c_2}{a}, \quad \text{hence} \quad a^2 = cc_2.$$

Adding the values of a^2 and b^2 just obtained, we finally get

$$a^2 + b^2 = cc_2 + cc_1 = c(c_1 + c_2) = c^2 \quad \text{because } c_1 + c_2 = c,$$

and this is the Pythagorean theorem. □

This second proof is not really shorter than Euclid's first (given in Section 2.5) when one takes into account the work needed to prove the proportionality of similar triangles. However, we often need similar triangles, so they are a standard tool, and a proof that uses standard tools is generally preferable to one that uses special machinery. Moreover, the splitting of a right-angled triangle into similar triangles is itself a useful tool—it enables us to construct the square root of any line segment.

Straightedge and compass construction of square roots

Given any line segment l, construct the semicircle with diameter $l+1$, and the perpendicular to the diameter where the segments 1 and l meet (Figure 2.19). Then *the length h of this perpendicular is $\sqrt{l}$.*

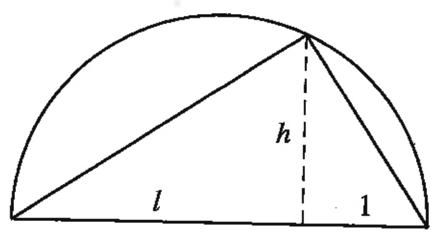

Figure 2.19: Construction of the square root

To see why, construct the right-angled triangle with hypotenuse $l+1$ and third vertex where the perpendicular meets the semicircle. We know that the perpendicular splits this triangle into two similar, and hence proportional, triangles. In the triangle on the left,

$$\frac{\text{long side}}{\text{short side}} = \frac{l}{h}.$$

In the triangle on the right,

$$\frac{\text{long side}}{\text{short side}} = \frac{h}{1}.$$

Because these ratios are equal by proportionality of the triangles, we have

$$\frac{l}{h} = \frac{h}{1},$$

hence $h^2 = l$; that is, $h = \sqrt{l}$. □

This result complements the constructions for the rational operations $+, -, \times$, and $\div$ we gave in Chapter 1. The constructibility of these and $\sqrt{}$ was first pointed out by Descartes in his book *Géométrie* of 1637. Rational operations and $\sqrt{}$ are in fact *precisely* what can be done with straightedge and compass. When we introduce coordinates in Chapter 3 we will see that any "constructible point" has coordinates obtainable from the unit length 1 by $+, -, \times, \div$, and $\sqrt{}$.

Exercises

Now that we know how to construct the $+, -, \times, \div$, and $\sqrt{}$ of given lengths, we can use algebra as a shortcut to decide whether certain figures are constructible by straightedge and compass. If we know that a certain figure is constructible from the length $(1+\sqrt{5})/2$, for example, then we know that the figure is constructible—period—because the length $(1+\sqrt{5})/2$ is built from the unit length by the operations $+, \times, \div$, and $\sqrt{}$.

This is precisely the case for the regular pentagon, which was constructed by Euclid in Book IV, Proposition 11, using virtually all of the geometry he had developed up to that point. We also need nearly everything we have developed up to this point, but it fills less space than four books of the *Elements*!

The following exercises refer to the regular pentagon of side 1 shown in Figure 2.20 and its diagonals of length x.

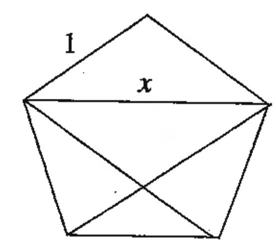

Figure 2.20: The regular pentagon

2.8.1 Use the symmetry of the regular pentagon to find similar triangles implying

$$\frac{x}{1} = \frac{1}{x-1},$$

that is, $x^2 - x - 1 = 0$.

2.8.2 By finding the positive root of this quadratic equation, show that each diagonal has length $x = (1+\sqrt{5})/2$.

2.8.3 Now show that the regular pentagon is constructible.