MIDTERM I   Friday, Jan. 29   NAME: Key

DIRECTIONS:

- This exam is a closed book, closed notes. Nothing should be on the table except a pencil, an eraser, and this exam.
- NO CALCULATORS or other PAPERS allowed.
- Show all work, clearly and in order.
- When required, **do not forget the units**!
- Circle your final answers. **You will loose points if you do not circle your answers.**
- This test has 10 problems, 2 extra credit problems, and 11 pages, It is your responsibility to make sure that you have all of the pages!

Question	Points	Score
1	10	
2	10	
3	10	
4	6	
5	12	
6	12	
7	10	
8	10	
9	10	
10	10	
Extra Credit 1	5	
Extra Credit 2	5	
Total	100	

Problem 1: (10 points) Consider the vectors $\mathbf{u} = \mathbf{i} + \mathbf{j} - \mathbf{k}$ and $\mathbf{v} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$.

(a) (5 points) Find the orthogonal projection of $\mathbf{u}$ on $\mathbf{v}$.

$$\text{proj}_{\vec{v}}\vec{u} = \frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|^2}\vec{v} = \frac{(1-2-1)}{(1+4+1)}(\vec{i}-2\vec{j}+\vec{k}) = \boxed{-\frac{1}{3}(\vec{i}-2\vec{j}+\vec{k})}$$

+1 +2 +2

(b) (5 points) Find $\mathbf{u} \times \mathbf{v}$. What is the geometric interpretation of this?

$$\vec{u}\times\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & -1 \\ 1 & -2 & 1 \end{vmatrix} = \vec{i}(1-2) - \vec{j}(1+1) + \vec{k}(-2-1)$$

+2

$$= \boxed{-\vec{i}-2\vec{j}-3\vec{k}}$$

+2

The vector $\vec{u}\times\vec{v}$ is perpendicular to both $\vec{u}$ and $\vec{v}$. Also $\|\vec{u}\times\vec{v}\|$ is the area of the parallelogram with adjacent sides $\vec{u}$, and $\vec{v}$.

+1

Problem 2: (10 points) Find the equation of the plane that passes through the points $(0,0,0)$, $(2,0,-1)$, and $(0,4,-3)$.

+2

The equation of a plane is given by $Ax+By+Cz=D$ (*)
Let $P_1=(0,0,0)$, $P_2=(2,0,-1)$, and $P_3=(0,4,-3)$. Then all three points must satisfy (*), Hence

+3

$0=D$
$2A-C=D$ let $A=1$ $\Rightarrow$ $C=2$, $B=\frac{3}{2}$ (+1)
$4B-3C=D$

(since $Ax+By+Cz=0$ is determined up to a scalar multiple)

+4 $x+\frac{3}{2}y+2z=0$

Problem 3: (10 points) Find the distance to the point $(6,1,0)$ from the plane that passes through the origin that is perpendicular to $\mathbf{i}-2\mathbf{j}+\mathbf{k}$.

The plane Π: $1(x-0)-2(y-0)+1(z-0)=0$ (+3)
$x-2y+z=0$

Distance from Π to $\vec{X}_0=(6,1,0)$ is

$$D=\frac{|Ax_0+By_0+Cz_0+D|}{[A^2+B^2+C^2]^{\frac{1}{2}}}=\frac{|6-2+0|}{\sqrt{6}}=\frac{4}{\sqrt{6}}$$

+3 +4

Problem 4: (6 points) Using an $\epsilon - \delta$ proof, show

$$lim_{(x,y)\to(0,0)}\frac{x^2}{\sqrt{x^2+y^2}} = 0.$$

(Since this is a proof, there is no need to circle your answer to problem 5).

Let $\varepsilon > 0$ be given. We want to find $\delta > 0$ such that

+1 $\|(x,y)-(0,0)\| < \delta \Rightarrow \left\|\frac{x^2}{\sqrt{x^2+y^2}} - 0\right\| < \varepsilon.$

+2 +2

$$\text{Consider } \left\|\frac{x^2}{\sqrt{x^2+y^2}} - 0\right\| = \left|\frac{x^2}{\sqrt{x^2+y^2}}\right| = \frac{x^2}{\sqrt{x^2+y^2}} \leq \frac{x^2+y^2}{\sqrt{x^2+y^2}} = \frac{\sqrt{x^2+y^2}^{\,2}}{\cancel{\sqrt{x^2+y^2}}}$$

since $\frac{x^2}{\sqrt{x^2+y^2}} > 0$

$$= \sqrt{x^2+y^2} = \|(x,y)-(0,0)\| < \delta$$

so choose $\delta = \varepsilon$. +1

Q.E.D.

Problem 5: (12 points) Consider the function $f(x, y) = e^{x+y}cos(xy)$ Find the equation of the plane tangent to the graph at the point $\mathbf{x}_0 = (0, 1)$.

The equation of the tangent plane is

(+1) $z = f(\vec{x}_0) + \frac{\partial f}{\partial x}\Big|_{\vec{x}_0}(x - x_0) + \frac{\partial f}{\partial y}\Big|_{\vec{x}_0}(y - y_0)$, $f(\vec{x}_0) = e^{0+1}\cos(0) = e$

(+2) $\frac{\partial f}{\partial x} = e^{x+y}\cos(xy) + e^{x+y}(-y)\sin(xy) = e^{x+y}[\cos(xy) - y\sin(xy)]$

(+2) $\frac{\partial f}{\partial x}\Big|_{\vec{x}_0} = e^{0+1}[\cos(0) - 1\cancel{\sin(0)}^{0}] = e$

(+2) $\frac{\partial f}{\partial y} = e^{x+y}\cos(xy) + e^{x+y}(-x)\sin(xy) = e^{x+y}[\cos(xy) - x\sin(xy)]$

(+2) $\frac{\partial f}{\partial y}\Big|_{\vec{x}_0} = e^{0+1}[\cos(0) - 0] = e$

$z = \cancel{e} + e(x - 0) + e(y - \cancel{1}) \Rightarrow$ $z = ex + ey$ (+3)

Problem 6 (12 points) Find the second order Taylor approximation of $f(x, y) = e^{x+y}cos(xy)$ at $\mathbf{x}_0 = (0, 1)$.

$\frac{\partial^2 f}{\partial x^2} = e^{x+y}[\cos(xy) - y\sin(xy)] + e^{x+y}[-y\sin(xy) - y^2\cos(xy)]$

(+3) $\frac{\partial^2 f}{\partial x^2}\Big|_{\vec{x}_0} = e^{0+1}[\cos(0) - 1\cancel{\sin(0)}^{0}] + e^{0+1}[-1\cancel{\sin(0)}^{0} - 1^2\cos(0)] = \cancel{e} - \cancel{e} = 0$

$\frac{\partial^2 f}{\partial y^2} = e^{x+y}[\cos(xy) - x\sin(xy)] + e^{x+y}[-x\sin(xy) - x^2\cos(xy)]$

(+3) $\frac{\partial^2 f}{\partial y^2}\Big|_{\vec{x}_0} = e^{0+1}[\cos(0) - 0] + e^{0+1}[0] = e$

(+3) $\frac{\partial^2 f}{\partial x \partial y} = e^{x+y}[\cos(xy) - x\sin(xy)] + e^{x+y}[-y\sin(xy) - \sin(xy) - xy\cos(xy)]$

$\frac{\partial^2 f}{\partial x \partial y}\Big|_{\vec{x}_0} = e^{0+1}[\cos(0) - 0] + e^{0+1}[-1\cancel{\sin(0)}^{0} - \cancel{\sin(0)}^{0} - 0] = e$

(+1) $z = e + eh_1 + eh_2 + \frac{1}{2}(\cancel{0h_1^2} + eh_2^2 + \cancel{2}eh_1h_2) + R_2(\vec{x}_0, \vec{h})$

(+2) $z = e[h_1 + h_2 + \frac{1}{2}h_2^2 + h_1h_2] + R_2(\vec{x}_0, \vec{h})$

Problem 7: (10 points) Given $\mathbf{g}(x,y) = (x^2+1, y^2)$ and $\mathbf{f}(u,v) = (u+v, u, v^2)$ and $\mathbf{x}_0 = (1,1)$,

(a) (3 points) Compute $\mathbf{Dg}(\mathbf{x}_0)$.

(+1) $D\vec{g} = \begin{pmatrix} 2x & 0 \\ 0 & 2y \end{pmatrix}$

(+2) $D\vec{g}(\vec{x}_0) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

(b) (4 points) Compute $\mathbf{Df}(\mathbf{u}_0)$ (remember $\mathbf{u}_0 = \mathbf{g}(\mathbf{x}_0)$).

$\vec{u}_0 = \vec{g}(\vec{x}_0) = (1+1, 1) = (2,1)$ (+1)

(+1) $D\vec{f} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 2v \end{pmatrix}$

(+2) $D\vec{f}(\vec{u}_0) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 2 \end{pmatrix}$

(c) (3 points) Compute $\mathbf{D}(\mathbf{f} \circ \mathbf{g})(\mathbf{x}_0)$.

$D(f\circ g)(\vec{x}_0) = D\vec{f}(\vec{u}_0) \cdot D\vec{g}(\vec{x}_0) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 2 & 0 \\ 0 & 4 \end{pmatrix}$

(+1) (+2)

Problem 8 (10 points) Consider the function $V = -\frac{GmM}{||\mathbf{r}||}$, where $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ (where x, y, and z are measured in meters), which is the gravitational potential of a mass m at (x, y, z) produced by gravitational force of a mass M centered at the origin. Note, G is a gravitational constant with units $\frac{\mathrm{m}^3}{\mathrm{kg}\cdot\mathrm{s}^2}$.

(a) (8 points) Compute ∇V.

(+1) $V = -GmM\left(\frac{1}{(x^2+y^2+z^2)^{\frac{1}{2}}}\right)$

(+3) $\nabla V = -GmM\left(-\frac{1}{\cancel{2}}\cdot \cancel{2}x(x^2+y^2+z^2)^{-\frac{3}{2}}, -\frac{1}{\cancel{2}}\cancel{2}y(x^2+y^2+z^2)^{-\frac{3}{2}}, -\frac{1}{\cancel{2}}\cancel{2}z(x^2+y^2+z^2)^{-\frac{3}{2}}\right)$

(+4) $= +GmM\left(\frac{x}{||\vec{r}||^3}, \frac{y}{||\vec{r}||^3}, \frac{z}{||\vec{r}||^3}\right)$

(b) (2 points) What is the interpretation of $-\nabla V$? (Hint: Think about the units. Think about what happens as you move in the direction of $-\nabla V$.)

(+1) $\left[GmM\frac{\vec{r}}{||\vec{r}||^3}\right] = \frac{\cancel{m^3}}{\cancel{kg}\cdot s^2}\cdot kg^{\cancel{2}}\cdot\frac{m}{\cancel{m^3}} = \frac{kg\cdot m}{s^2} = N$ (i.e. force)

(+1) The gradient gives the direction of greatest increase. Because the units of $-\nabla V$ are N, we are looking at the direction we must travel to increase the force $\cdot$ m (potential) the greatest. Because $-\nabla V \propto +\frac{\vec{r}}{||\vec{r}||^3}$ this means it is always pointing out from the origin. Hence, the farther from mass M, we get, the greater the gravitational potential.

Problem 9 (10 points) Consider the function $f(x,y) = x^2 + y^2 - xy$

(a) (5 points) Find the critical points of f.

+1 $\frac{\partial f}{\partial x} = 2x - y \overset{set}{=} 0 \Rightarrow y = 2x$

+1 $\frac{\partial f}{\partial y} = 2y - x \overset{set}{=} 0 \Rightarrow y = \frac{x}{2}$

+1 $2x = \frac{x}{2} \Rightarrow x = 0 \Rightarrow y = 0$

So the critical point is given by $(0,0)$

+2

(b) (5 points) Use the second derivative test to determine wether the critical points are local maxima, minima, or saddle points.

+1 $\frac{\partial^2 f}{\partial x^2} = 2 > 0$, $\frac{\partial^2 f}{\partial y^2} = 2$ +1

+1 $\frac{\partial^2 f}{\partial x \partial y} = -1 \Rightarrow D = \left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right) - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = 4 - 1 = 3 > 0$ +1

So $(0,0)$ is a minimum

+1

Problem 10 (10 points) Consider the function $f(x, y) = x$ subject to the constraint $x^2 + 2y^2 = 3$.

(a) (5 points) Find the critical points of f.

Consider $g(x,y) = x^2+2y^2-3$, need solutions to

+1 $\nabla f = \lambda \nabla g$

$1 = 2\lambda x$

$0 = 4\lambda y \Rightarrow \lambda = 0$ or $y=0$ +2

$x^2+2y^2=3$

Case 1: $\lambda = 0$ $\ast$

+1 Case 2: $y=0 \Rightarrow x^2=3 \Rightarrow x = \pm\sqrt{3} \Rightarrow \lambda = \pm\frac{1}{2\sqrt{3}}$

+1 So the critical points are $(\sqrt{3}, 0)$ and $(-\sqrt{3}, 0)$

(b) (5 points) Use the bordered Hessian to classify the critical points.

+1 Consider $h(x,y) = f(x,y) - \lambda[g(x,y)] = x - \lambda(x^2+2y^2-3) = 0$

$$|\bar{H}| = \begin{vmatrix} 0 & -2x & -4y \\ -2x & 2\lambda & 0 \\ -4y & 0 & 4\lambda \end{vmatrix}$$

+1 +1

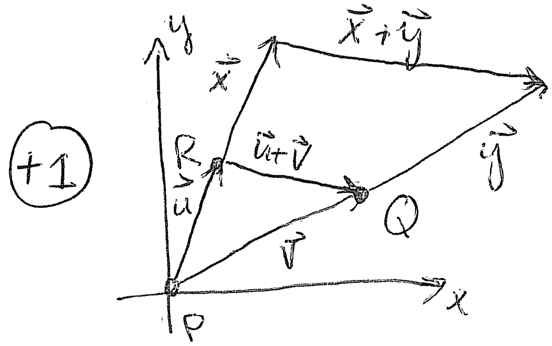
$$|\bar{H}|\Big|_{(\sqrt{3},0,\frac{1}{2\sqrt{3}})} = \begin{vmatrix} 0 & -2\sqrt{3} & 0 \\ -2\sqrt{3} & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & \frac{2}{\sqrt{3}} \end{vmatrix} = \frac{2}{\sqrt{3}}(4\sqrt{3}^2) = 8\sqrt{3} > 0 \Rightarrow \text{Maxima}$$

+1 +1

$$|\bar{H}|_{(-\sqrt{3},0,-\frac{1}{2\sqrt{3}})} = \begin{vmatrix} 0 & 2\sqrt{3} & 0 \\ 2\sqrt{3} & -\frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & -\frac{2}{\sqrt{3}} \end{vmatrix} = -\frac{2}{\sqrt{3}}(4\sqrt{3}^2) = -8\sqrt{3} < 0 \Rightarrow \text{Minima}$$

Extra Credit 1 (5 points) Suppose PQR is a triangle in space and $b > 0$ is a number. Prove that there exists a triangle with sides parallel to those of PQR and lengths b times those of PQR. (Since this is a proof, there is no need to circle your answer to this problem.)

+1



Let P = origin and let $\vec{u} = \overline{PR}$ and $\vec{v} = \overline{PQ}$ +1

Then the third side $\overline{RQ} = \vec{u} + \vec{v}$

+1

Consider $b > 0$ given. Then $\vec{x} = b\vec{u}$ is parallel to $\vec{u}$ and $\vec{y} = b\vec{v}$ is parallel to $\vec{v}$

and $\vec{x} + \vec{y} = b(\vec{u} + \vec{v})$ is parallel to $\vec{u} + \vec{v}$. Also, the sides

+1 of the new triangle formed by $\vec{x}$, $\vec{y}$, and $\vec{x}+\vec{y}$ have sides parallel to PQR with lengths b times those of PQR since $\|\vec{x}\| = b\|\vec{u}\|$,

$\|\vec{y}\| = b\|\vec{v}\|$ and

$\|\vec{x}+\vec{y}\| = b\|\vec{x}+\vec{y}\|$.

Q.E.D.

Extra Credit 2 (5 points) Show that near the point $(x, y, u, v) = (1, 1, 1, 1)$ we can solve

$$\begin{aligned} xu + yvu^2 &= 2, \\ xu^3 + y^2v^4 &= 2, \end{aligned}$$

uniquely for u and v as functions of x and y. (Since this is a proof, there is no need to circle your answer to this problem.)

(+1) Let $F_1(x,y,u,v) = xu + yvu^2 - 2$,

$F_2(x,y,u,v) = xu^3 + y^2v^4 - 2$. and let $\vec{x}_0 = (1,1,1,1)$

(+1) Then By the Implicit Function thm., if we have $F_i(\vec{x}_0) = 0$ and $\Delta\big|_{\vec{x}_0} \neq 0$, we can represent u and v as functions of x and y

Clearly $F_1(\vec{x}_0) = 0$ and $F_2(\vec{x}_0) = 0$

(+2) $$\Delta = \begin{vmatrix} \frac{\partial F_1}{\partial u} & \frac{\partial F_1}{\partial v} \\ \frac{\partial F_2}{\partial u} & \frac{\partial F_2}{\partial v} \end{vmatrix} = \begin{vmatrix} x+2yvu & yu^2 \\ 3xu^2 & 4y^2v^3 \end{vmatrix}$$

$$\Delta\Big|_{\vec{x}_0} = \begin{vmatrix} 3 & 1 \\ 3 & 4 \end{vmatrix} = 12 - 3 = 9 \neq 0$$ so solvability is assured.

(+1)