MIDTERM II **Monday, Feb. 22** **NAME:** Key

DIRECTIONS:

- This exam is a closed book, closed notes. Nothing should be on the table except a pencil, an eraser, and this exam.
- NO CALCULATORS allowed.
- Show all work, clearly and in order.
- When required, **do not forget the units!**
- Circle your final answers. **You will lose points if you do not circle your answers.**
- This test has 8 problems, 1 extra credit problem, and 11 pages, It is your responsibility to make sure that you have all of the pages!

Question	Points	Score
1	15	
2	10	
3	15	
4	10	
5	10	
6	15	
7	15	
8	10	
Extra Credit 1	5	
Total	100	

Problem 1: (15 points) Consider the path given by $\mathbf{c}(t) = t\sin t\mathbf{i} + t\cos t\mathbf{j} + t^2\mathbf{k}$.

(a) (3 points) Calculate the velocity at time $t = 0$.

$\vec{v}(t) = \vec{c}\,'(t) = (\sin t + t\cos t)\vec{i} + (\cos t - t\sin t)\vec{j} + 2t\vec{k}$

$\vec{v}(0) = \vec{c}\,'(0) = 0\cdot\vec{i} + 1\cdot\vec{j} + 0\cdot\vec{k} = (0, 1, 0)$

(b) (3 points) Calculate the acceleration at time $t = 0$.

$\vec{a}(t) = \vec{c}\,''(t) = (2\cos t - t\sin t)\vec{i} + (-2\sin t - t\cos t)\vec{j} + 2\vec{k}$

$\vec{a}(0) = \vec{c}\,''(0) = (2, 0, 2)$

(c) (3 points) Calculate the tangent line at time $t = 0$.

$l(t) = \vec{c}(0) + \vec{v}(0)t = (0,0,0) + (0,1,0)t$

$\vec{c}(0) = (0,0,0)$

so $l(t) = (0,0,0) + (0,1,0)t$

(d) (6 points) Calculate the arclength of $\mathbf{c}(t)$ between $t = 0$ and $t = 2$.

$$\text{Arclength} = L(\vec{c}(t)) = \int_0^2 \|\vec{c}\,'(t)\|\,dt$$

from part (a), $\vec{v}(t) = \vec{c}\,'(t) = \left((\sin t + t\cos t), (\cos t - t\sin t), 2t\right)$

$$\|\vec{v}(t)\| = \left(\underline{\sin^2 t} + \cancel{2t\sin t\cos t} + t^2\cos^2 t + \underline{\cos^2 t} - \cancel{2t\sin t\cos t} + t^2\sin^2 t + 4t^2\right)^{\frac{1}{2}}$$

$$= (1 + 5t^2)^{\frac{1}{2}}$$

so

$$L(\vec{c}(t)) = \int_0^2 \sqrt{1 + (\sqrt{5}t)^2}\,dt \qquad \text{let } u = \sqrt{5}t \Rightarrow du = \sqrt{5}\,dt$$

$$t = 0 \to u = 0, \quad t = 2 \to u = 2\sqrt{5}$$

$$= \frac{1}{\sqrt{5}}\int_0^{2\sqrt{5}} \sqrt{1+u^2}\,du = \frac{1}{2\sqrt{5}}\left[u\sqrt{u^2+1} + 1^2\log\left|u + \sqrt{u^2+1}\right|\right]\Bigg|_0^{2\sqrt{5}}$$

from (*) on page 11

$$= \frac{1}{2\sqrt{5}}\left[2\sqrt{5}\sqrt{21} + \log\left|2\sqrt{5} + \sqrt{21}\right|\right] - \cancel{\frac{1}{2\sqrt{5}}(0)}^{\,0}$$

$$= \sqrt{21} + \frac{1}{2\sqrt{5}}\log\left(2\sqrt{5} + \sqrt{21}\right)$$

Problem 2: (10 points) Suppose that the speed of a given path, $||\mathbf{c}(t)||$ is constant. Prove that $\mathbf{c}'(t)$ is perpendicular to $\mathbf{c}(t)$. (Since this is a proof, there is no need to circle your answer.)

<u>Prove</u> $\vec{c}(t) \circ \vec{c}\,'(t) = 0$

<u>Proof:</u> We know that $||\vec{c}(t)|| = k$, where k is constant.

$||\vec{c}(t)||^2 = \vec{c}(t) \cdot \vec{c}(t) = k^2$ still constant.

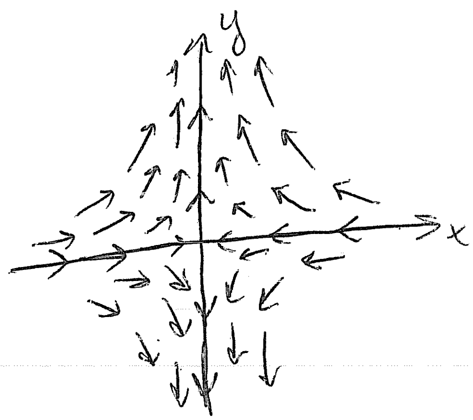
Taking the derivative of both sides

$$2\vec{c}(t) \cdot \vec{c}\,'(t) = 0 \Rightarrow \underbrace{\vec{c}(t) \circ \vec{c}\,'(t) = 0}_{\text{Hence } \vec{c}(t) \perp \vec{c}\,'(t)}$$

Q.E.D.

Problem 3: (15 points) Consider the vector field given by $\mathbf{F} = (-x, y)$.

(a) (6 points) Sketch the vector field in the xy-plane. (Since this is a sketch, there is no need to circle your answer.)



(b) (6 points) What is the divergence of this vector field?

$$\operatorname{div} \vec{F} = \vec{\nabla} \cdot \vec{F} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot (-x, y, 0) = -1 + 1 + 0 = 0$$

(c) (3 points) What is the physical meaning of divergence?

The divergence tells you if, and to what degree, a volume "carried" by the vector field is compressed or expanded by the "flow".

Problem 4: (10 points) Verify that the path $\mathbf{c}(t) = (\sin t, \cos t, e^{2t})$ is a flow line of the vector field $\mathbf{F} = (y, -x, 2z)$.

Need to show $\vec{c}\,'(t) = \vec{F}(\vec{c}(t))$

$\vec{c}\,'(t) = (\cos t, -\sin t, 2e^{2t})$
$F(\vec{c}(t)) = (\cos t, -\sin t, 2e^{2t})$

hence we know $\vec{c}(t)$ is a flow line of the vector field $\vec{F}$.

Problem 5: (10 points) Show that $\mathbf{F} = (x^2 + y^2)\,\mathbf{i} - 2xy\mathbf{j}$ is not a gradient vector field. (Since this is a proof, there is no need to circle your answer.)

Want to show $\vec{\nabla} \times \vec{F} \neq 0$

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x^2+y^2) & -2xy & 0 \end{vmatrix} = \vec{i}(0) - \vec{j}(0) + \vec{k}(-2y - 2y)$$

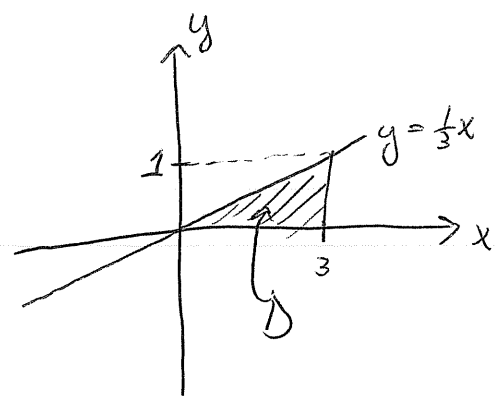
$$= -4y\vec{k} \neq 0$$

Hence, $\vec{F}$ cannot be a gradient vector field.

Problem 6 (15 points) Evaluate the following.

$$\int_0^1 \int_{3y}^3 \cos\left(x^2\right)\, dx dy.$$

We begin by changing the order of Integration



$$\iint_D \cos(x^2)\, dA = \int_0^3 \int_0^{x/3} \cos(x^2)\, dy dx$$

$$= \int_0^3 y\cos(x^2) \Big|_0^{x/3} dx = \frac{1}{3}\int_0^3 x\cos(x^2)\, dx$$

let $u = x^2$, $du = 2x\, dx$

$x = 0 \Rightarrow u = 0$, $x = 3 \Rightarrow u = 9$

$$= \frac{1}{6}\int_0^9 \cos(u)\, du = \frac{1}{6}\sin u \Big|_0^9 = \frac{1}{6}\sin(9)$$

Problem 7: (15 points) Calculate the volume of the region bounded by the planes $x = 0$, $y = 0$, $z = 0$, and $x + y + z = 1$.

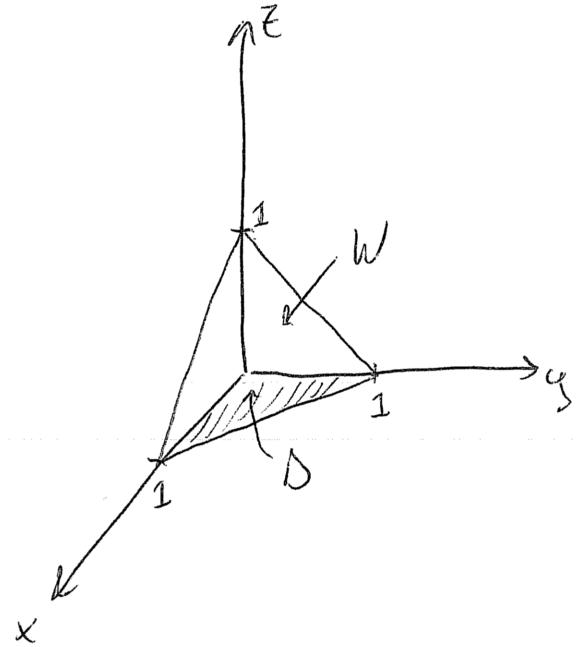
$$\text{Volume} = \iiint_W dV = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} dz\,dy\,dx$$

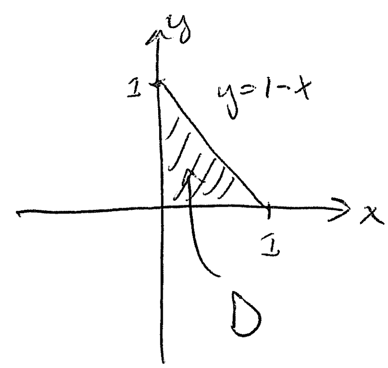
$$= \int_0^1 \int_0^{1-x} (1-x-y)\,dy\,dx$$

$$= \int_0^1 (1-x)y - \frac{y^2}{2} \Big|_0^{1-x} dx$$

$$= \frac{1}{2}\int_0^1 (1-x)^2\,dx = -\frac{1}{2}(1-x)^3\left(\frac{1}{3}\right)\Big|_0^1$$

$$= \frac{1}{6}$$





Problem 8 (10 points)

(a) (8 points) Use the Mean Value Inequality to show that

$$\frac{1}{\sqrt{3}} \leq \int\int_D \frac{1}{\sqrt{1+x^6+y^8}}dxdy \leq 1.$$

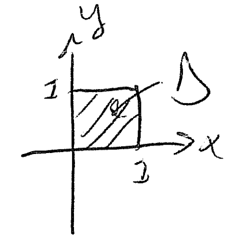
Where $D = [0,1] \times [0,1]$. (Since this is a proof, there is no need to circle your answer.)

Let $f(x,y) = \frac{1}{\sqrt{1+x^6+y^8}}$, f is continuous on D and therefore attains its minimum, m, and maximum, M, at some points (x_1,y_1) and $(x_2,y_2) \in D$. Note $f(x,y) \geq 0$ $\forall (x,y) \in D$

$m = \min(f) = \frac{1}{\sqrt{3}}$ occurs when the denom. is largest (at $(1,1)$)

$M = \max(f) = 1$ occurs when the denom. is smallest (at $(0,0)$)

The area of D, $A(D) = 1 \times 1 = 1$, hence the mean value inequality yields



$$\frac{1}{\sqrt{3}} \cdot 1 \leq \iint_D f\,dA \leq 1 \cdot 1$$

Q.E.D.

(b) (2 points) Why would you ever want to use this theorem? Why not just always calculate the integral directly?

Sometimes it is difficult (or inconvenient) to calculate the value of the integral directly. Additionally, there are times when "getting a handle" on the size or bounding of the integral is enough or informative in its own right.

Extra Credit 1 (5 points) Sow that the path $\mathbf{c}(t) = t\cos t\mathbf{i} + t\sin t\mathbf{j} + t\mathbf{k}$ lies on the surface of the cone $z^2 = x^2 + y^2$.

We want to show that if $\vec{c}(t) = (c_1, c_2, c_3)$

$c_3^2 = c_1^2 + c_2^2$.

$$\begin{cases} c_3^2 = t^2 \\ c_1^2 + c_2^2 = t^2\cos^2 t + t^2\sin^2 t = t^2(\sin^2 t + \cos^2 t) = t^2 \end{cases}$$

$\rightarrow$ Hence, $\vec{c}(t)$ lies on the surface of the cone $z^2 = x^2 + y^2$

INTEGRATION FORMULAS: Note, just because a formula is given here, does not mean that is necessary for any of the given problems. Use these as needed only.

$$\int \frac{\sqrt{a+bx}}{x}dx = 2\sqrt{a+bx} + a\int \frac{1}{x\sqrt{a+bx}}dx$$

$$\int \frac{1}{x\sqrt{a+bx}}dx = \frac{2(3bx-2a)(a+bx)^{3/2}}{15b^2}$$

$$(*)\int \sqrt{x^2 \pm a^2}dx = \frac{1}{2}\left[x\sqrt{x^2 \pm a^2} + a^2 \log\left|x + \sqrt{x^2 \pm a^2}\right|\right]$$

$$\int \frac{1}{\sqrt{a^2-x^2}}dx = \sin^{-1}\left(\frac{x}{a}\right) \quad (a > 0)$$

$$\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\tan\left(\frac{x}{a}\right) \quad (a > 0)$$

$$\int \frac{1}{\sqrt{a^2+x^2}}dx = \log\left(x + \sqrt{a^2+x^2}\right) = \sinh^{-1}\left(\frac{x}{a}\right) \quad (a > 0)$$