p-Coloring Classes of Torus Knots

Anna-Lisa Breiland
Department of Mathematics
Willamette College
Salem, OR 97301
abreilan@willamette.edu

Layla Oesper
Department of Mathematics and Computer Science
Pomona College
Claremont, CA 91711
layla.oesper@pomona.edu

Laura Taalman
Department of Mathematics and Statistics
James Madison University
Harrisonburg, VA 22807
taal@math.jmu.edu

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Abstract

We develop a theorem for determining the p-colorability of any (m, n) torus knot. We also prove that any p-colorable (m, n) torus knot has exactly one p-coloring class. Finally, we show that every p-coloring of the braid projection of an (m, n) torus knot must use all of the p colors.

1 Introduction

This paper has two main results; both of which are proved from algebraic and geometric perspectives. The first result is a theorem specifically determining the p-colorability of any (m, n) torus knot. This result has been highly investigated and proven in limited subcases. It has been shown that a $(m, m-1)$ torus knot is always p-colorable for p equal to m or $m-1$ depending on which is odd [6] [14]. Another proven result is that a $(2, n)$ torus knot is always p-colorable for p equal to n and a $(3, n)$ torus knot is always 3-colorable if n is even [14]. A result similar to ours was also stated as a lemma without proof in [4].

Our second major result shows that any p-colorable (m, n) torus knot has only one p-coloring class. The number of p-coloring classes for a knot is an invariant under the Reidemeister moves. A general result investigating colorings of torus knots by finite Alexander quandles appears in [3]. Our result is a special instance of this result, however, we use only simple techniques while the proof of the more general result involves more complicated strategies. p-coloring classes have also been previously been investigated in relationship to pretzel knots by [5].

Using our second main result, we were also able to show a minor result concerning the distribution of colors in a p-coloring of a torus knot. We showed that any p-coloring of the braid representation of an (m, n) torus knot must use each of the p colors. Distribution of colors in p-colorings of knots has been previously investigated with the Kauffman-Harary Conjecture. This conjecture is concerned with the distribution of colors in a Fox p-coloring of an alternating knot with prime determinant. Asaeda, Przytcki, and Sikora prove the Harary-Kauffman Conjecture is true for pretzel knots and Montesinos knots in [2].

In Section 2 we present basic definitions of knots, p-colorability, p-coloring classes, p-nullity, and torus knots. Even though p-colorability can be defined for any integer $p \geq 2$, in this paper we will only address cases where p is an odd prime.

In Section 3 we discuss an algebraic method for finding the determinant, and hence the p-colorability, of any (m, n) torus knot. We then use braid representations to examine this idea geometrically.

In Section 4we address the p-coloring classes of torus knots. We thoroughly examine this topic and its relationship to torus knots, finally determining that all torus knots have a single p-coloring class. This is done using both algebraic and geometric methods. The geometric method gives rise to an additional result concerning the distribution of colors in any p-coloring of a torus knot.

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2 Definitions and Background Information

We begin by presenting some basic definitions and background pertaining to knots, colorability, and the class of torus knots. See also [1] and [10].

2.1 p-colorability and the dihedral group

One of the main goals of knot theory is to be able to distinguish between all knots. One way to accomplish this goal is by examining a knot's p-colorability as seen in Definition 1.

Definition 1. *Given a prime number $p > 2$ we say that a projection of a knot K is* p-colorable *if every strand in the projection can be labeled using numbers 0 to $p-1$, with at least 2 of the labels distinct, so that at each crossing we have*

$$2x - y - z = 0 \bmod p, \tag{1}$$

where x is the overstrand and y and z are the understrands of the crossing.

Since p-colorability is preserved by Reidemeister moves, it is a knot invariant. Hence, we say that a knot K is *p-colorable* if any projection of K is p-colorable. A 3-colorable knot is called *tricolorable*. In this case the condition in Equation (1) is equivalent to having all strands the same color, or all strands different colors, at each of the crossings of K [10]. An example of a 3-coloring is shown in Figure 1.

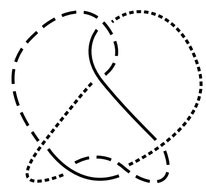

Figure 1: A 3-coloring of the knot 6_1

The *Wirtinger presentation* of a knot K is obtained by labeling the strands of an oriented projection of K with $s_1, s_2, \cdots, s_m$. Maintaining the right hand rule, place a loop x_i around each strand s_i such that the loop begins and ends at a base point in the knot complement. In Figures 2 and 3 the loop x_i begins at a base point above the figure, travels to the tail of the arrow representing x_i, and follows that arrow beneath strand s_i before returning to it origin above the figure. The set of loops $\{x_1, x_2, \cdots, x_n\} = Q$ makes up the Wirtinger presentation for K. The loops surrounding any crossing of K will have one of the two following relations:

$$x_j^{-1} x_k x_j x_i^{-1} = 1 \text{ (see Figure 2)}$$

and therefore at such a crossing we have

$$x_j^{-1} x_k x_j = x_i.$$

Or,

$$x_j^{-1} x_i x_j x_k^{-1} = 1 \text{ (see Figure 3)}$$

and therefore at such a crossing we have

$$x_j^{-1} x_i x_j = x_k.$$

The difference between these two relations is due entirely to choice of orientation. Therefore, any crossing where x_j is the loop around the overstrand and x_i, x_k are the loops around the understrands gives rise to the relation $x_j^{-1} x_i x_j = x_k$ [13].

It can be shown that the p-colorability condition in Equation (1) comes from the relation between the dihedral group

$$D_{2p} = \langle a, b \mid a^p = 1 = b^2, b^{-1}ab = a^{-1} \rangle$$

and the knot group

$$\pi_0(S^3 - K) = \langle x_1, x_2, \cdots, x_m \in Q \mid x_j^{-1} x_i x_j = x_k \text{at each crossing of } K \rangle,$$

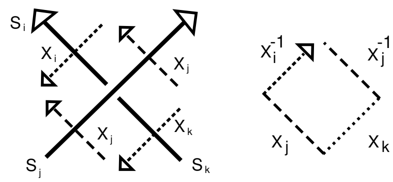

Figure 2: Wirtinger loop near a right handed crossing.

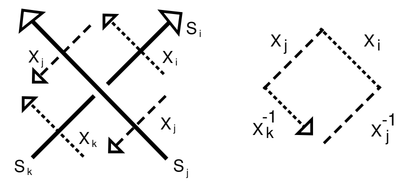

Figure 3: Wirtinger loop near a left handed crossing.

where Q is the Wirtinger presentation of the knot.

Theorem 1. *Let K be an oriented knot. Then K is p-colorable if and only if the map*

$$\theta\colon \pi_0(S^3 - K) \to D_{2p}$$

defined by $\theta(x_i) = ba^{i \bmod p}$ is a well-defined homomorphism.

Proof. ($\Rightarrow$) Given that K is p-colorable, we must show that θ is well-defined in the sense that it respects the relations of the knot group. In the exponents of the computations below we will refer to $i \bmod p$ simply as i. If x_i, x_j, x_k surround a crossing of K and x_j is the loop around the overstrand, we have:

$$\begin{aligned}
\theta(x_i^{-1}x_jx_i) &= \theta(x_i)^{-1}\theta(x_j)\theta(x_i) && (\theta \text{ is a homomorphism})\\
&= (ba^i)^{-1}ba^jba^i && (\text{definition of } \theta)\\
&= a^{-i}b^{-1}ba^jba^i &&\\
&= a^{j-i}ba^i &&\\
&= ba^{2i-j} && (\text{relation in } D_{2p})\\
&= ba^k && (p\text{-coloring condition})\\
&= \theta(x_k). &&
\end{aligned}$$

($\Leftarrow$) We will show that the conditions that make θ a well-defined homomorphism imply the p-colorability conditions. In the exponents of the computations below we will refer to $i \bmod p$ simply as i.

$$\begin{aligned}
\theta(x_i^{-1}x_jx_i) = \theta(x_k) &\Longrightarrow (ba^i)^{-1}ba^jba^i = ba^k\\
&\Longrightarrow a^{-i}b^{-1}ba^jba^i = ba^k\\
&\Longrightarrow a^{j-i}ba^i = ba^k\\
&\Longrightarrow a^{j-i}ba^ib^{-1} = ba^kb^{-1}\\
&\Longrightarrow a^{j-2i} = a^{-k}\\
&\Longrightarrow 2i - j - k = 0 \bmod p.
\end{aligned}$$

□

2.2 p-coloring classes

Let $G_p(K)$ be the set of all p-colorings for a knot K. Note that $G_p(K)$ is empty if K is not p-colorable. We wish to count the number of p-colorings in $G_p(K)$ that differ by more than just a permutation of the colors. To do this precisely we will redefine p-colorability so that we can partition $G_p(K)$ with an equivalence relation. The following definition is clearly equivalent to Definition 1.

Definition 2. *Suppose $\mathcal{S}_K$ is the set of all strands of K. A p-*coloring *of a knot K is a map*

$$\gamma : \mathcal{S}_K \to \mathbb{Z}_p$$

satisfying the condition that

$$2\gamma(s_j) - \gamma(s_i) - \gamma(s_k) = 0 \bmod p$$

for all $s_i, s_j, s_k \in \mathcal{S}_k$ at a crossing of K, where s_j is the overcrossing strand and s_i, s_k are the undercrossing strands.

For all $\gamma, \delta \in G_p(K)$, consider the relation $\sim$ defined by

$$\gamma \sim \delta \Longleftrightarrow \gamma = \rho \circ \delta \text{ for some permutation } \rho \colon \mathbb{Z}_p \to \mathbb{Z}_p. \tag{2}$$

Theorem 2. *The relation $\sim$ defined in Equation (2) is an equivalence relation on $G_p(K)$.*

Proof. Suppose γ, δ, and α are p-colorings in $G_p(K)$. We have three things to show about the relation $\sim$:

Reflexive: Define ρ to be the identity map, which is obviously one-to-one and onto and thus a permutation of $\mathbb{Z}_p$. Then $\gamma = \rho \circ \gamma$ for all $\gamma \in G_p(K)$. Hence, $\gamma \sim \gamma$ for all $\gamma \in G_p(K)$.

Symmetric: Assume that $\gamma \sim \delta$. Then there exists a permutation ρ such that $\gamma = \rho \circ \delta$. Therefore $\rho^{-1} \circ \gamma = \delta$. Since ρ is one-to-one and onto, so is ρ^{-1}. Therefore we have $\delta \sim \gamma$.

Transitive: Assume that $\gamma \sim \delta$ and $\delta \sim \alpha$. Thus there exist permutations ρ and ρ' such that $\gamma = \rho \circ \delta$ and $\delta = \rho' \circ \alpha$. Therefore $\gamma = \rho \circ (\rho' \circ \alpha) = (\rho \circ \rho') \circ \alpha$. Since ρ and ρ' are both one-to-one and onto, their composition is one-to-one and onto as well. Therefore we have $\gamma \sim \alpha$. $\square$

Definition 3. *The p-*coloring class *of $\gamma \in G_p(K)$ is the set $\bar{\gamma} = \{\delta \in G_p | \delta \sim \gamma\}$. Two p-colorings are said to be* equivalent *if they are in the same p-coloring class, and* fundamentally different *if they are in different p-coloring classes.*

Definition 4. *The set of p-coloring classes for a given knot K will be denoted by $C_p(K)$. The number of p-coloring classes for K will be denoted by $|C_p(K)|$.*

For example, in Figure 4 the first two 3-colorings of knot 9_{35} are equivalent. In the same figure the third 3-coloring of 9_{35} is fundamentally different from each of the first two colorings (and therefore $|C_3(9_{35})|$ is at least 2).

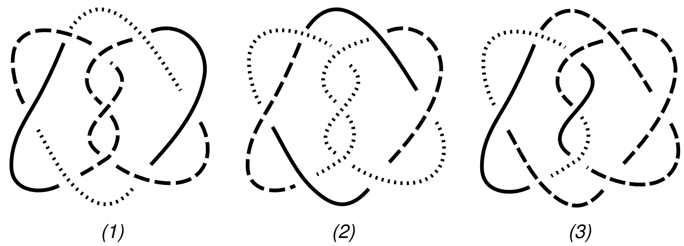

Figure 4: Colorings (1) and (2) of the 9_{35} knot are equivalent, while coloring (3) is fundamentally different from colorings (1) and (2).

2.3 Determinants, crossing matrices, and p-nullity

A p-coloring of a knot projection K determines a linear equation at each crossing of K. The matrix for the resulting system of linear equations is called the *crossing matrix* of K. Any one row and column can be eliminated from the crossing matrix in order to create a *minor crossing matrix.* Every minor crossing matrix of any projection of K will have the same determinant (up to sign). The absolute value of this determinant is what we call the *determinant* of the knot [10]. The determinant of a knot K completely determines the prime numbers p for which K is p-colorable. (Although non-prime p can be considered, in this paper we will only consider prime p-colorings.)

Theorem 3. *Suppose p is a prime number. A knot K is p-colorable if and only if p divides* $\det(K)$.

For a proof of Theorem 3 see [10]. Notice that a knot is p-colorable for some prime p if and only if $\det(K) \neq 1$.

Definition 5. *The* p-nullity *of a knot K is the dimension of the null space of its associated crossing matrix modulo p.*

It should be noted that in some papers, p-nullity is defined as the dimension of the nullity of the minor crossing matrix but in this paper we will be referring to the nullity of the complete crossing matrix.

It can be shown that p^r gives the number of different ways to assign p colors to a knot K according the rules of p-colorings, including the p trivial colorings. Thus $p^r - p$ is the number of non-trivial p-colorings [10].

Theorem 4. *Given a knot K with p-nullity r, the number of different, but not necessarily fundamentally different, p-colorings of K is given by $p^r - p$ [10].*

In fact, the number of fundamentally different p-colorings of a knot K can be discerned from $K's$ p-nullity. It is clear that a knot has p-nullity 1 if and only if it has only the trivial colorings [5]. As p-nullity of K increases, $|C_p(K)|$ also increases (see Table 1 and Table 2). A specific equation relating p-nullity and $|C_p(K)|$ can be found in [5].

Theorem 5 is obtained directly from Tables 1 and 2.

p-nullity	$\lvert C_p(K)\rvert$
1	0
2	1
3	4
4	14
5	51
6	202

Table 1: The number of p-coloring classes for a given p-nullity r (the table holds for any p provided $p \geq r$).

3-nullity	$\lvert C_3(K)\rvert$
1	0
2	1
3	4
4	13
5	40
6	121

Table 2: The number of 3-coloring classes for a 3-nullity r knot (note the tables differ only when $p < r$).

Theorem 5. *A knot K has p-nullity 2 if and only if $|C_p(K)| = 1$. [5]*

For Example, using Figure 4 we previously determined that $|C_3(9_{35})| \geq 2$. By computing the nullity of the crossing matrix for 9_{35} and using Table 2 it can be easily shown that $|C_3(9_{35})| = 4$ (see [5]).

2.4 Torus knots

All mathematical knots can be broken into three disjoint groups: torus, satellite and hyperbolic. In this paper we will be examining p-colorings of torus knots. A *torus knot* is a knot that lies on an unknotted torus without crossing over or under itself as it lies on the torus; for example, see the knot in Figure 5.

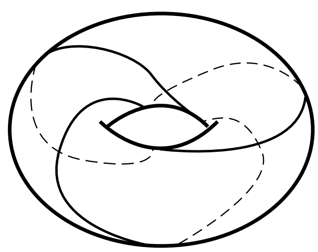

Figure 5: Trefoil knot on a torus.

A torus knot $T_{m,n}$ is completely characterized by the number of times m that it circles around the meridian of the torus and the number of times n that it circles around the longitude of the torus. For example, the trefoil knot in Figure 5 is the torus knot $T_{3,2}$. A torus knot $T_{m,n}$ is a knot and not a link if and only if m and n are relatively prime. The torus knot $T_{m,n}$ is equivalent to the torus knot $T_{n,m}$ [1].

A *braid* is a set of n strings which are attached to a horizontal bar at the top and the bottom. Each string in a braid must always "head downwards";

in other words, each string will intersect a horizontal plane exactly once. If we connect each of the strands on the top bar with the corresponding strands on the bottom bar we obtain a knot. It is known that every knot can be represented by a braid, called the *braid representation* of the knot [1]. For example, the trefoil knot shown in Figure 5 has the braid representation shown in Figure 6.

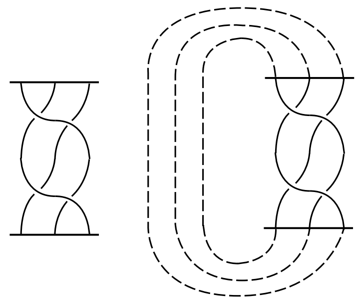

Figure 6: The braid representation of a trefoil knot.

The *braid word* for a braid is a description of the projection of the braid when the projection is arranged so that no two crossings occur at the same height. For every crossing where strand i crosses strand $i+1$, the crossing is written as σ_i if the crossing is positive (i.e. if σ_i crosses over σ_{i+1}) and σ_i^{-1} if the crossing is negative (i.e. if σ_{i+1} crosses over σ_i). For example, the braid representation of the knot 6_1 in Figure 7 has braid word $\sigma_3^2\sigma_2\sigma_3^{-1}\sigma_1^{-1}\sigma_2\sigma_1^{-1}$.

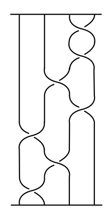

Figure 7: The braid representation of 6_1.

The torus knot $T_{m,n}$ can be drawn as the closure of the n-strand braid word

$$(\sigma_1\sigma_2\cdots\sigma_{n-2}\sigma_{n-1})^m.$$

We will refer to the word $(\sigma_1\sigma_2\cdots\sigma_{n-2}\sigma_{n-1})$ as the *base* for the braid word of $T_{m,n}$. Figure 6 shows the braid representation of the (2,3) torus knot, also known as the trefoil. The braid presented in this figure has the full braid word $(\sigma_1\sigma_2)^2$. The base for this braid word is $\sigma_1\sigma_2$.

A *cycle* is a single completion of the base for the braid word of a knot $T_{m,n}$, as shown in Figure 8. For the duration of the paper, any braid representation of a knot $T_{m,n}$ is considered to have n strands and m cycles. We will also consider the 0^{th} strand to be the strand on the far left of the braid representation.

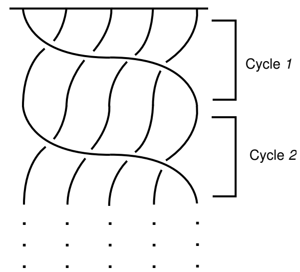

Figure 8: Two cycles of a braid.

3 p-Colorability of Torus Knots

Our first main result completely characterizes the p-colorability of torus knots. Note that if $T_{m,n}$ is a torus knot, then m and n are relatively prime, and thus m and n cannot both be even.

Main Theorem 1. *Suppose $T_{m,n}$ is a torus knot and p is prime.*

i) If m and n are both odd, then $T_{m,n}$ is not p-colorable.

ii) If m is odd and n is even, then $T_{m,n}$ is p-colorable if and only if $p|m$.

iii) If m is even and n is odd, then $T_{m,n}$ is p-colorable if and only if $p|n$.

In Section 3.1 we will prove this theorem from an algebraic perspective, by using the Jones polynomial to find $\det(T_{m,n})$. In Section 3.2 we will revisit Main Theorem 1 from a more geometric perspective, and exhibit a particular p-coloring for every p-colorable torus knot.

Results similar to those in Main Theorem 1 were stated without proof by Asami and Satoh in [4].

3.1 Using determinants to determine p-colorability

In this section we will prove Main Theorem 1 using the following theorem, which describes the determinant of any torus knot $T_{m,n}$.

Theorem 6. *Given any torus knot $T_{m,n}$, we have*

$$\det(T_{m,n}) = \begin{cases} 1, & \text{if } m \text{ and } n \text{ are both odd} \\ m, & \text{if } m \text{ is odd and } n \text{ is even} \\ n, & \text{if } m \text{ is even and } n \text{ is odd.} \end{cases}$$

Main Theorem 1 will follow immediately from Theorem 6 and Theorem 3.

Our proof of Theorem 6 will use knot polynomials. Given a knot K, the *Alexander polynomial* is constructed using the orientation of K and manipulating the crossings of a projection of K in to solve a basic equation. Another polynomial, the *Jones polynomial*, is formed by looking at the projection of K and manipulating each crossing using a series of Skein relations [1].

Proof. Given a knot K, it is well known that $\det(K) = |\Delta_K(-1)|$, where $\Delta_K(t)$ is the Alexander polynomial of K (see [11]). It is also known that if $V_K(t)$ is the Jones polynomial of K, then $\Delta_K(-1) = V_K(-1)$ (see [8]). Therefore we know that for any knot K, $\det(K) = |V_K(-1)|$.

The equation for the Jones polynomial of the torus knot $T_{m,n}$ is (see [9])

$$V_{T_{m,n}}(t) = \frac{t^{(m-1)(n-1)/2}(t^{m+n} - t^{n+1} - t^{m+1} + 1)}{(1-t^2)}. \tag{3}$$

To find $|V_{T_{m,n}}(-1)|$ we must first reduce the expression in Equation (3). Clearly $t = -1$ is always a root of $t^{m+n} - t^{n+1} - t^{m+1} + 1$ regardless of the parity of m and n (as long as they are not both even). We will use synthetic division to remove a factor of $t+1$ from this polynomial. Without loss of generality we will assume $m > n$. Depending on the parity of m and n, there are three cases.

Case 1. Suppose m and n are both odd. The synthetic division in this case is as follows. (The labels at the top indicate the degree $m+n$, $m+1$, and $n+1$ coefficients.)

$$\begin{array}{r|rrrrrrrrrrrrrrrr}
 & \scriptstyle(m+n) & & & & & \scriptstyle(m+1) & & & & & \scriptstyle(n+1) & & & & & \\
 & 1 & 0 & 0 & \cdots & 0 & -1 & 0 & 0 & \cdots & 0 & -1 & 0 & 0 & \cdots & 0 & 1 \\
-1 & & -1 & 1 & \cdots & -1 & 1 & 0 & 0 & \cdots & 0 & 0 & 1 & -1 & \cdots & 1 & -1 \\
\hline
 & 1 & -1 & 1 & \cdots & -1 & 0 & 0 & 0 & \cdots & 0 & -1 & 1 & -1 & \cdots & 1 & \boxed{0} \\
 & \multicolumn{5}{c}{\underbrace{\hphantom{1\ -1\ 1\ \cdots\ -1}}_{\text{even}}} & & & & & & \multicolumn{5}{c}{\underbrace{\hphantom{-1\ 1\ -1\ \cdots\ 1}}_{\text{even}}} &
\end{array}$$

This leaves us with the polynomial

$$(t^{m+n-1} - t^{m+n-2} + \cdots - t^{m+1}) + (-t^n + t^{n-1} - \cdots + 1). \tag{4}$$

Therefore, the Jones polynomial is now

$$\frac{t^{(m-1)(n-1)/2}[(t^{m+n-1} - t^{m+n-2} + \cdots - t^{m+1}) + (-t^n + t^{n-1} - \cdots + 1)]}{(1-t)}.$$

By substituting -1 for t (keeping in mind that m and n are both odd) we get

$$\det(T_{m,n}) = |V_{T_{m,n}}(-1)| = \left|\frac{(\pm 1)[(-1)(n-1) + (1)(n+1)]}{2}\right| = 1.$$

Case 2. Suppose m is even and n is odd. The only difference between this case and the first case is that the difference between $m+1$ and $n+1$ is odd in this case. However, this difference only determines the number of zeros in the middle block of the synthetic division calculation above, so we obtain the same quotient as in Equation (4), and the same form of Jones polynomial. In this case, however, when we substitute -1 for t we get

$$\det(T_{m,n}) = |V_{T_{m,n}}| = \left|\frac{(\pm 1)[(1)(n-1) + (1)(n+1)]}{2}\right| = n.$$

Case 3. Suppose m is odd and n is even. In this case, the synthetic division is as follows:

$$
\begin{array}{r|rrrrrrrrrrrrrrrr}
 & {\scriptstyle (m+n)} & & & & & {\scriptstyle (m+1)} & & & & & {\scriptstyle (n+1)} & & & & & \\
 & 1 & 0 & 0 & \cdots & 0 & -1 & 0 & 0 & \cdots & 0 & -1 & 0 & 0 & \cdots & 0 & 1 \\
-1 & & -1 & 1 & \cdots & 1 & -1 & 2 & -2 & \cdots & -2 & 2 & -1 & 1 & \cdots & 1 & -1 \\
\hline
 & 1 & -1 & 1 & \cdots & 1 & -2 & 2 & -2 & \cdots & -2 & 1 & -1 & 1 & \cdots & 1 & \boxed{0}
\end{array}
$$

$$
\underbrace{1 \; -1 \; 1 \; \cdots \; 1}_{\text{odd}} \qquad \underbrace{-2 \; 2 \; -2 \; \cdots \; -2}_{\text{odd}} \qquad \underbrace{1 \; -1 \; 1 \; \cdots \; 1}_{\text{odd}}
$$

By completing our synthetic division we will obtain the polynomial

$$(t^{m+n-1}-t^{m+n-2}+\cdots+t^{m+1})+(-2t^{m}+2t^{m-1}-\cdots-2t^{n+1})+(t^{n}-t^{n-1}+\cdots+1).$$

When we substitute -1 in for t in this polynomial we get

$$(1)(n-1)+(2)(m-n)+(1)(n+1)=2m.$$

Therefore the determinant in this case is

$$\det(T_{m,n}) = |V_{T_{m,n}}(-1)| = \left|\frac{(\pm 1)(2m)}{2}\right| = m.$$

$\square$

3.2 Using braid representations to determine p-colorability

The determinant argument in Section 3 proves Main Theorem 1, but in this section we present another proof from a geometric perspective. Some of the techniques in this subsection will be particularly useful in Section 4 when we investigate the number of p-coloring classes of torus knots and in addition these methods set the stage for our results in Corollary 8.

Before we begin our proof, we need to build up a basic framework of definitions and properties.

Definition 6. *Given a p-colored braid representation of a knot K, the j^{th}* color array *of K is the element of $(\mathbb{Z}_p)^n$ whose i^{th} component is the color of the i^{th} strand of the braid representation of K after j cycles.*

For a knot K to be p-colorable it is a necessary and sufficient condition that the initial color array of its braid representation be exactly the same as its final color array.

Let $(c_0, c_1, \cdots, c_{n-1})$ be the j^{th} color array for the braid representation of some knot K, and consider the map $\phi : (\mathbb{Z}_p)^n \to (\mathbb{Z}_p)^n$ defined by

$$\phi(c_0, c_1, \cdots, c_{n-1}) = (2c_0 - c_1, 2c_0 - c_2, \cdots, 2c_0 - c_{n-1}, c_0). \tag{5}$$

Notice that ϕ is the map that, given the j^{th} color array of a knot K, returns the $(j+1)^{\text{st}}$ color array according the rules of p-colorability, as seen in Figure 9.

We also define the map ϕ^j to be the composition of j copies of ϕ. Notice that $\phi^j : (\mathbb{Z}_p)^n \to (\mathbb{Z}_p)^n$ is then the map defined by

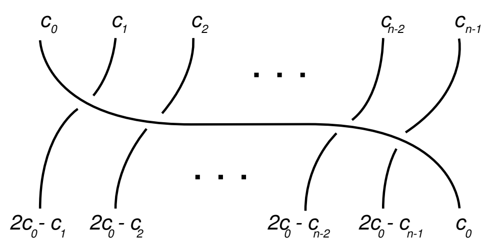

Figure 9: The action of ϕ on the j^{th} color array of $T_{m,n}$.

$$\phi^j(\text{ initial color array of } K) = j^{\text{th}} \text{ color array of } K.$$

A p-coloring of any knot K is entirely determined by the initial color array of a braid representation for K. Furthermore, for a knot K to be p-colorable, it is a necessary and sufficient condition that we have $\phi^m = \text{id}$ when applied to the initial color array of the braid representation.

We now consider a second n-tuple that can be defined from the braid representation of a p-colored knot K. The *color variance* between any two adjacent strands in the projection colored with c_i and c_j respectively is $c_j - c_i \bmod p$. (Note: We consider the far left and far right strands to be adjacent.)

Definition 7. *Given a p-colored braid representation of a knot K, the j^{th}* variance vector of $T_{m,n}$ *is the element of $(\mathbb{Z}_p)^n$ whose i^{th} component is the color variance between the $(i-1)^{\text{th}}$ and i^{th} strands after j cycles.*

The 0^{th} variance vector of K is referred to as the *initial variance vector*. A *constant variance vector* is a variance vector $V = (v_0, v_1, \cdots, v_{n-1})$ where $v_0 = v_1 = \cdots = v_{n-1}$. If $(c_0, c_1, \cdots, c_{n-2}, c_{n-1})$ is the j^{th} color array for some knot $T_{m,n}$, then the j^{th} variance vector for $T_{m,n}$ is

$$(v_0, v_1, \cdots, v_{n-2}, v_{n-1}) = (c_1 - c_0, c_2 - c_1, \cdots, c_{n-1} - c_{n-2}, c_0 - c_{n-1}). \quad (6)$$

We will let $\psi\colon (\mathbb{Z}_p)^n \to (\mathbb{Z}_p)^n$ denote the map that takes as input the j^{th} variance vector of a knot K and returns the $(j+1)^{\text{st}}$ variance vector. By Equation (5) and (6) we have:

$$\begin{aligned}\psi(v_0, v_1, \cdots, v_{n-2}, v_{n-1}) &= ((2c_0 - c_2) - (2c_0 - c_1), (2c_0 - c_3) - (2c_0 - c_2), \\ &\quad \cdots, c_0 - (2c_0 - c_{n-1}), (2c_0 - c_1) - c_0 \\ &= (c_1 - c_2, c_2 - c_3, \cdots, c_{n-1} - c_0, c_0 - c_1) \\ &= (-v_1, -v_2, \cdots, -v_{n-1}, -v_0).\end{aligned}$$

Figure 10 shows how Equation (7) looks as we move from the j^{th} to the $(j+1)^{\text{th}}$ variance vector. In this figure, the color variance v_i between the $(i-1)^{\text{st}}$ and i^{th} strands is shown between those two strands.

We define $\psi^j : (\mathbb{Z}_p)^n \to (\mathbb{Z}_p)^n$ to be the composition of j copies of ψ. Since for every application of ψ to $(v_0, v_1, \cdots, v_{n-1})$, all entries get multiplied by -1

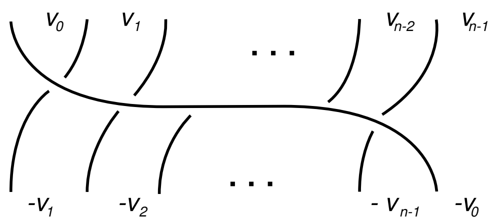

Figure 10: The action of ψ on the j^{th} variance vector of $T_{m,n}$.

and move over one position to the left while wrapping around, we have that $\psi^j(v_0, v_1, \cdots, v_{n-1})$ acts such that for all $i \in \{0, 1, \cdots, n-1\}$

$$v_i \longrightarrow (-1)^j v_{i+j \bmod n}. \tag{7}$$

In a more general form this means that

$$\psi^j(v_0, v_1, \cdots, v_{n-1}) = \begin{cases} (v_j, v_{j+1}, \cdots, v_0, v_1, \cdots, v_{j-1}), & \text{if } j \text{ is even} \\ (-v_j, -v_{j+1}, \cdots, -v_0, -v_1, \cdots, -v_{j-1}), & \text{if } j \text{ is odd,} \end{cases}$$

where all subscripts are taken modulo n.

We now have the basic definitions we need to give an alternate proof of Main Theorem 1.

Proof. We will consider two cases. First, we will consider the case where m is even and n is odd (without loss of generality this will also prove the case where m is odd and n is even). In this case we will exhibit a specific p-coloring of $T_{m,n}$ for every prime p that divides n. Second, we will consider the case where both m and n are odd, and show that there is no p-coloring of $T_{m,n}$ for any prime number p.

Case 1. Suppose m is even and n is odd, and that p divides n. In order to show that $T_{m,n}$ is p-colorable we will exhibit a p-coloring of $T_{m,n}$ that we will call the *main p-coloring* of $T_{m,n}$.

Since p divides n there exists $r \in \mathbb{Z}^+$ such that $n = rp$. Consider the braid representation of $T_{m,n}$ with n strands and m cycles whose initial, or 0^{th}, color array M is the n-tuple whose entries are the series $0, 1, \cdots, p-2, p-1$, repeated r times:

$$M = (0, 1, \cdots, p-2, p-1, 0, 1, \cdots, p-1, \cdots, p-2, p-1). \tag{8}$$

Using Equation (5) we can deduce that the 1^{st} color array of $T_{m,n}$ is

$$\begin{aligned} \phi(M) &= \phi(0, 1, \cdots, p-2, p-1, 0, 1, \cdots, p-1, \cdots, p-2, p-1) \\ &= (2(0)-1, 2(0)-2, \cdots, 2(0)-(p-1), 2(0)-0, \\ &\qquad 2(0)-1, 2(0)-2, \cdots, 2(0)-0, \cdots, 2(0)-(p-1), 0) \\ &= (-1, -2, \cdots, -(p-1), 0, -1, -2, \cdots, 0, \cdots, -(p-1), 0) \\ &= (p-1, p-2, \cdots, 1, 0, p-1, p-2, \cdots, 0, \cdots, 1, 0). \end{aligned}$$

Similarly, the 2^{nd} color array of $T_{m,n}$ is

$$
\begin{aligned}
&\phi^2(M)\\
&\quad = \phi(\phi(M))\\
&\quad = \phi(p-1, p-2, \cdots, 1, 0, p-1, p-2, \cdots, 0, \cdots, 1, 0)\\
&\quad = (2(p-1)-(p-2), 2(p-1)-(p-3), \cdots, 2(p-1)-0,\\
&\qquad\qquad 2(p-1)-(p-1), 2(p-1)-(p-2), 2(p-1)-(p-3), \cdots,\\
&\qquad\qquad 2(p-1)-(p-1), \cdots, 2(p-1)-0, (p-1))\\
&\quad = (p, p+1, \cdots, 2p-2, 2p-1, p, p+1, \cdots, 2p-1, \cdots, 2p-2, p-1)\\
&\quad = (0, 1, \cdots, p-2, p-1, 0, 1, \cdots, p-1, \cdots, p-2, p-1)\\
&\quad = M.
\end{aligned}
$$

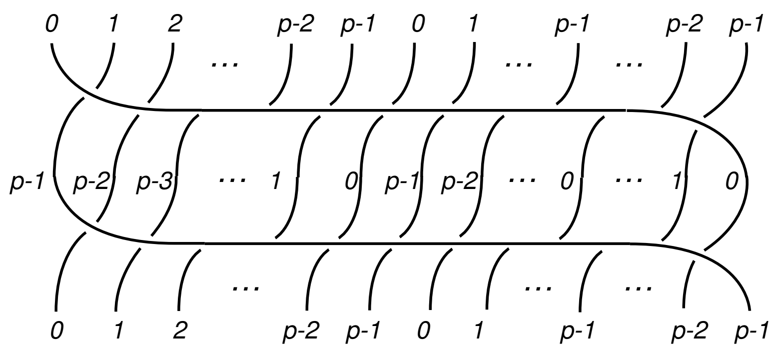

Figure 11: The action of ϕ on M.

See Figure 11 for an illustration of the action of ϕ and ϕ^2. Since $\phi^2(M) = M$ and m is even, we can conclude that $\phi^m(M) = M$. Since our initial and final color arrays are the same, we have a p-coloring of the torus knot $T_{m,n}$.

Case 2. Suppose m and n are both odd. If we can show that a braid representation of a knot $T_{m,n}$ is not p-colorable, then we will have shown that $T_{m,n}$ is not p-colorable. Seeking a contradiction we will assume that $T_{m,n}$ is p-colorable and then show that its only coloring is the trivial one.

Let $C = (c_0, c_1, \cdots, c_{n-1})$ be an initial color array for some p-coloring of $T_{m,n}$. Let $V = (v_0, v_1, \cdots, v_{n-1})$ be the variance vector associated with C. Since C induces a p-coloring of $T_{m,n}$, we have

$$\psi^m(v_0, v_1, \cdots, v_{n-1}) = (v_0, v_1, \cdots, v_{n-1}).$$

Using Equation (7), we can conclude that for any $k \in \mathbb{Z}^+$, and any $i \in \{0, 1, \cdots, n-1\}$ we have

$$v_i = (-1)^{km} v_{i+km \bmod n}. \tag{9}$$

In particular, when $k = n$ we have

$$
\begin{aligned}
v_i &= (-1)^{nm} v_{i+nm \bmod n} && \text{(Equation(9))} \\
&= (-1)^{nm} v_i && (i + nm \bmod n = i \bmod n) \\
&= -v_i && (n \text{ and } m \text{ are odd}).
\end{aligned}
$$

But this implies that $v_i = -v_i$ for all $i \in \{0, 1, \cdots, n-1\}$. Since $v_i \in \mathbb{Z}_p$ and p is prime (thus odd), this implies that v_i must equal 0 for all i. This means that V is a constant variance vector with variance 0. Hence, C would induce a trivial p-coloring on $T_{m,n}$. Therefore, our assumption that $T_{m,n}$ is p-colorable must be incorrect. Thus, $T_{m,n}$ is not p-colorable. $\square$

4 Counting p-Coloring Classes of Torus Knots

Our second main result describes the number of p-coloring classes of any torus knot $T_{m,n}$.

Main Theorem 2. *If p is prime and the torus knot $T_{m,n}$ is p-colorable, then $|C_p(T_{m,n})| = 1$.*

In Section 4.1 we will prove this theorem using braid representations and the variance vectors introduced in Section 3.2. We also prove a corollary of this result that is concerned with the distribution of colors in a p-coloring of $T_{m,n}$. In Section 4.2 we will revisit this theorem in the context of crossing matrices and p-nullity.

Main Theorem 2is a special case of a result found by Asami and Kuga in [3]. They prove that if a knot $T_{m,n}$ can be p-colored using a finite Alexander quandle, it has a total of p^2 trivial and non-trivial colorings. If $T_{m,n}$ cannot be colored by such a quandle, then it has only the p trivial colorings. It is important to note that Asami and Kuga only consider the total number of all p-colorings without distinguishing between equivalent colorings (as in Theorem 4, while we consider equivalence classes of p-colorings.

The distribution of colors used to p-color a knot has been previously examined by Asaeda, Przytychi and Sikora in [2]. They investigated the Kauffman-Harary Conjecture in relation to pretzel knots and Montesinos knots. Their results show that given a knot K of one of these types which is alternating with prime determinant, any p-coloring of K assigns different colors to different strands. Hence, any color used in a p-coloring of K is only used once. As we will see in Corollary 8, the methods of Section 4.1 will show that all p colors must be used.

4.1 Using braid representations to find $|C_p(T_{m,n})|$

Given a torus knot $T_{m,n}$, consider two p-colorings $\alpha, \beta \in G_p(T_{m,n})$. We know α and β are in the same coloring class in $C(T_{m,n})$ if $\alpha \sim \beta$ as in Equation (2). Using the braid projection of $T_{m,n}$ we can be more specific. Let α_i^j and β_i^j be the colors of the i^{th} strand of any braid knot K after j cycles for the p-colorings α and β, respectively. The condition for $\alpha \sim \beta$ from Equation (2) is clearly equivalent to the condition

$$\alpha_i^j = \alpha_l^k \iff \beta_i^j = \beta_l^k \tag{10}$$

for all $i, l \in \{0, 1, \ldots, n-1\}$ and for all $j, k \in \{0, 1, \ldots, m\}$.

To prove Main Theorem 2 we must show that if $T_{m,n}$ is p-colorable, then every p-coloring of $T_{m,n}$ is equivalent to the main p-coloring of $T_{m,n}$ that we defined in Section 3.2 (see Equation (8)). Notice that the initial variance vector for the main p-coloring of a knot $T_{m,n}$ as defined in Equation (8) is the constant variance vector $(1, 1, \cdots, 1)$. Our first step in proving Main Theorem 2 will be to prove the following theorem.

Theorem 7. *Suppose $T_{m,n}$ is a p-colorable torus knot where p is prime, and consider the n-strand braid representation of $T_{m,n}$. If the initial color array of the p-coloring has a constant variance vector, then that initial color array induces a p-coloring that is equivalent to the main p-coloring of $T_{m,n}$.*

Proof. Let C be an initial color array of $T_{m,n}$ with constant variance vector $(v, v, \ldots, v)$. The for some $a \in \mathbb{Z}_p$ we can write C as

$$C = (a + 0v, a + v, \cdots, a + (n-2)v, a + (n-1)v). \tag{11}$$

The first color array of $T_{m,n}$ with this p-coloring is

$$\begin{aligned}
\phi(C) &= \phi(a, a + v, \cdots, a + (n-2)v, a + (n-1)v) \\
&= (2a - (a + v), 2a - (a + 2v), \cdots, 2a - (a + (n-1)v), a) \\
&= (a - v, a - 2v, \cdots, a - (n-1)v, a) \\
&= (a + (n-1)v, a + (n-2)v, \cdots, a + v, a). \qquad (12)
\end{aligned}$$

Notice that ϕ reverses the order of the entries in C. Applying ϕ again, we see that the second color array is

$$\begin{aligned}
&\phi^2(C) \\
&\quad = \phi(a - v, a - 2v, \cdots, a - (n-1)v, a) \\
&\quad = (2(a - v) - (a - 2v), 2(a - v) - (a - 3v), \cdots, 2(a - v) - a, a - v) \\
&\quad = (a, a + v, \cdots, a + (n-3)v, a - 2v, a - v) notag \qquad (13) \\
&\quad = (a, a + v, \cdots, a + (n-3)v, a + (n-2)v, a + (n-1)v) \\
&\quad = C. \qquad (14)
\end{aligned}$$

Thus we can see that $\phi^2 = \text{id}$ when applied to color arrays with constant variance. (Notice the similarity between this proof and the geometric proof of Main Theorem 1.)

Since p is prime, there exists $t \in \mathbb{Z}_p$ such that $a = tv \bmod p$. Thus, $C = (tv, tv{+}v, tv{+}2v, \cdots, tv{+}(n{-}1)v) = (tv, (t{+}1)v, (t{+}2)v, \cdots, (t{+}(n{-}1))v)$. We know if $v \in \mathbb{Z}_p/\{0\}$ and p is prime, then $\langle v \rangle = \mathbb{Z}_p$. Therefore the first p entries of C are distinct elements of $\mathbb{Z}_p$. Moreover, since p divides n, C is comprised of this exact pattern of all of $\mathbb{Z}_p$ repeated precisely n/p times. Therefore,

$$C = (c_0, c_1, \cdots, c_{p-2}, c_{p-1}, c_0, c_1, \cdots, c_{p-1}, \cdots, c_{p-2}, c_{p-1}) \tag{15}$$

where $c_0, c_1, \cdots, c_{p-1}$ repeats n/p times. Since $c_0, c_1, \cdots, c_{p-1}$ are distinct, we know that for $i \in \{0, 1, \cdots, p-1\}$ we have

$$c_i = c_j \iff i = j. \tag{16}$$

Using the notation in Equation (16), and the results in Equations (12) and (14), we see that the j^{th} color array for the p-coloring induced by the initial color array C is

$$\phi^j(C) = \begin{cases} (c_0, c_1, \cdots, c_{p-2}, c_{p-1}, c_0, c_1, \cdots, c_{p-1}, \cdots, c_{p-2}, c_{p-1}), & \text{if } j \text{ is even} \\ (c_{p-1}, c_{p-2}, \cdots, c_1, c_0, c_{p-1}, c_{p-2}, \cdots, c_0, \cdots, c_1, c_0), & \text{if } j \text{ is odd.} \end{cases}$$

On the other hand, from the geometric proof of Main Theorem 1 we know that the j^{th} color array for the main p-coloring induced by the initial color array M from Equation (8) is

$$\phi^j(M) = \begin{cases} (0, 1, \cdots, p-2, p-1, 0, 1, \cdots, p-1, \cdots, p-2, p-1), & \text{if } j \text{ is even} \\ (p-1, p-2, \cdots, 1, 0, p-1, p-2, \cdots, 0, \cdots, 1, 0), & \text{if } j \text{ is odd.} \end{cases}$$

We wish to prove that C and M induce equivalent p-colorings in terms of the condition in Equation (10). Let $\pi_i : (\mathbb{Z}_p)^n \to (\mathbb{Z}_p)$ be the i^{th} projection map, and let $\phi_i^j = \pi_i \circ \phi^j : (\mathbb{Z}_p)^n \to (\mathbb{Z}_p)$. Notice that ϕ_i^j takes in an initial color array and returns the color of the i^{th} strand of the j^{th} color array of the induced p-coloring. From the expression for $\phi^j(C)$ above, we see that for $j \in \{0, 1, \cdots, m\}$ and $i \in \{0, 1, \cdots, n-1\}$ we have

$$\phi_i^j(C) = \begin{cases} c_i \bmod p, & \text{if } j \text{ is even} \\ c_{p-1-i} \bmod p, & \text{if } j \text{ is odd.} \end{cases} \tag{17}$$

Similarly, from the expression for $\phi^j(M)$ above, we see that

$$\phi_i^j(M) = \begin{cases} i \bmod p, & \text{if } j \text{ is even} \\ p-1-i \bmod p, & \text{if } j \text{ is odd.} \end{cases} \tag{18}$$

It is now easy to see from Equations (16), (17), and (18) that we have

$$\phi_i^j(M) = \phi_l^k(M) \iff \phi_i^j(C) = \phi_l^k(C)$$

for $j, k \in \{0, 1, \cdots, m\}$ and $i, l \in \{0, 1, \cdots, n-1\}$. Therefore by Equation (10) we know that C and M induce equivalent p-colorings on $T_{m,n}$. □

We will now use Theorem 7 to prove Main Theorem 2. The key will be to show that there cannot be a p-coloring of $T_{m,n}$ that does not have a constant variance vector.

Proof. Suppose $T_{m,n}$ is a p-colorable torus knot, and consider the n-strand braid representation of $T_{m,n}$. By Main Theorem 1 we can assume without loss of generality that n is odd and m is even, and that p divides n.

Seeking a contradiction, assume that $|C_p(T_{m,n})| > 1$. Specifically, assume there is a p-coloring $\gamma \in G_p(T_{m,n})$ that is fundamentally different from the main p-coloring of $T_{m,n}$. Let $G = (g_0, g_1, \cdots, g_{n-2}, g_{n-1})$ be the initial color array for γ, and let $V = (v_0, v_1, \cdots, v_{n-2}, v_{n-1}) = (g_1 - g_0, g_2 - g_1, \cdots, g_{n-1} - g_{n-2}, g_0 - g_{n-1})$ be the initial variance vector for γ. Using ψ as defined in Equation (7) we can apply ψ^q to V to get the q^{th} variance vector of γ:

$$\psi^q(V) = \begin{cases} (v_q, v_{q+1}, \cdots, v_0, v_1, \cdots, v_{q-1}), & \text{if } q \text{ is even} \\ (-v_q, -v_{q+1}, \cdots, -v_0, -v_1, \cdots, -v_{q-1}), & \text{if } q \text{ is odd.} \end{cases}$$

In the above equation, all subscripts should be taken mod n.

Let r be the smallest positive integer for which V partitions into s repeating sections of length r. Note that $1 \leq r \leq n$ and that $n = rs$. Since by hypothesis γ is not equivalent to the main p-coloring of $T_{m,n}$, Theorem 7 tells us that the variance vector V is not constant. Therefore we have $r > 1$.

Since m is even, we know from Equation (5) that

$$\phi^m(V) = (v_m, v_{m+1}, \cdots, v_0, v_1, \cdots, v_{m-1}).$$

In other words, ϕ^m turns the initial variance vector V into the vector where all entries have been shifted to the left m positions and wrapped around. Because m is the number of cycles in our braid representation, and γ is a p-coloring of $T_{m,n}$, we must have $\psi^m(V) = V$. In order for this to occur, ψ^m must shift V over by some multiple of r, the length of a repeating section. Therefore we must have $m = kr$ for some $k \in \mathbb{Z}^+$.

We have now shown that $n = rs$, $m = kr$, and $r > 1$; these facts imply that $\gcd(m, n) \geq r > 1$. But this contradicts our assumption that $T_{m,n}$ is a knot and not a link. Therefore, there cannot exist a p-coloring γ that is fundamentally different from the main coloring, and hence we must have $|C_p(T_{m,n})| = 1$. □

Corollary 8. *Every p-coloring of a braid projection of knot $T_{m,n}$ must use all p colors.*

Proof. This follows directly from Main Theorems 1 and 2, since the main p-coloring of $T_{m,n}$ uses all p colors, and every p-coloring that is equivalent to the main coloring must also use all p colors. □

4.2 Using p-nullity to find $|C_p(T_{m,n})|$

We now revisit Main Theorem 2 from a linear algebra perspective. By Theorem 5 we can show that a knot K has only one p-coloring class by showing that K has p-nullity 2. Therefore Main Theorem 2 is equivalent to Theorem 9.

Theorem 9. *The p-nullity of any p-colorable torus knot $T_{m,n}$ is 2.*

By Main Theorem 1 we know that if $T_{m,n}$ is p-colorable then m and n have different parity. We will prove Theorem 9 in the case where the larger of m and n is odd. (The other case is left to a future paper, as it is less systematic than this case.) Our strategy will be to reduce the general crossing matrix for $T_{m,n}$ in this case. The notation of circulant matrices will simplify this process.

A *circulant matrix* is a square matrix of the form shown in Equation (19). The entries in each row of C are identical to the entries in the previous row, except that they are moved over one position to the right and wrapped around. Notice that a circulant matrix is completely determined by its first row [7].

$$C = \mathrm{Circ}(c_1, c_2, \cdots, c_n) = \begin{pmatrix} c_1 & c_2 & \cdots & c_n \\ c_n & c_1 & \cdots & c_{n-1} \\ \vdots & \vdots & & \vdots \\ c_2 & c_3 & \cdots & c_1 \end{pmatrix}. \tag{19}$$

With this notation we are ready to prove the special case of Theorem 9.

Proof. Without loss of generality we will assume throughout this proof that we have $m > n$. We also assume the special case where m is odd and n is even (see question (3) in Section 5. The choices of labelings of crossings and strands in Figures 12 and 13 induce a particularly nice crossing matrix M for $T_{m,n}$:

$$M = \begin{pmatrix} I & & & & & A \\ S & I & & & & T \\ & S & I & & & T \\ & & \ddots & \ddots & & \vdots \\ & & & S & I & T \\ & & & & S & B \end{pmatrix} \tag{20}$$

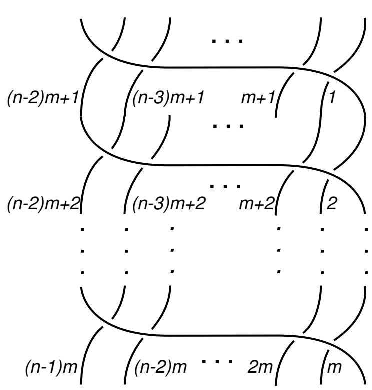

Figure 12: labeling of torus strands

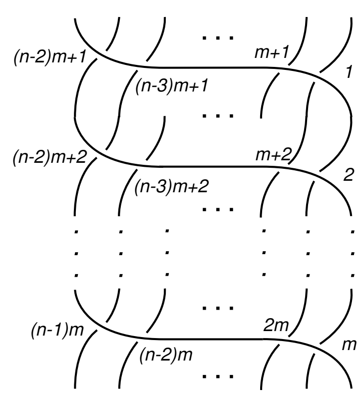

Figure 13: labeling of torus crossings

The matrix M in Equation (20) is an $m(n-1)\times m(n-1)$ matrix consisting of square blocks of size m. Specifically, I represents the $m\times m$ identity matrix, and S, T, A, and B represent the following $m\times m$ circulant matrices:

$$\begin{array}{rcl} S &=& \mathrm{Circ}(\begin{array}{ccccccc} 0 & 0 & \cdots & 0 & 0 & 1 \end{array}), \\ T &=& \mathrm{Circ}(\begin{array}{ccccccc} 0 & 0 & \cdots & 0 & 0 & -2 \end{array}), \\ A &=& \mathrm{Circ}(\begin{array}{ccccccc} 0 & 0 & \cdots & 0 & 1 & -2 \end{array}), \\ B &=& \mathrm{Circ}(\begin{array}{ccccccc} 1 & 0 & \cdots & 0 & 0 & -2 \end{array}). \end{array}$$

In order to determine the p-nullity of M we will reduce M to upper triangular form. By looking at the expressions for M and S, we see that row R_{m+1} is the first row of M whose leading 1 is not on the diagonal. Our first sequence of row operations will reduce M to a matrix whose upper left square block of size $(n-1)m-1$ is the identity matrix. For $m+1\le k\le (n-1)m-1$, we reduce as follows:

$$R_k = \begin{cases} R_k - R_{k-1} & \text{if } k = 1 \bmod m \\ R_k - R_{k-m-1} & \text{if } k \neq 1 \bmod m \end{cases} \tag{21}$$

The sequence of reductions in (21) results in the following reduced matrix:

$$M' = \begin{pmatrix} I & & & & K_1 \\ & I & & & K_2 \\ & & \ddots & & \vdots \\ & & & I & K_{n-2} \\ & & & & R^T \end{pmatrix}$$

In this matrix, the form of each $m\times m$ block K_i depends on whether i is even or odd, as follows:

$$K_i = \begin{cases} \mathrm{Circ}(\begin{array}{ccccccc} 0 & \cdots & 0 & 1 & -2 & 2 & \cdots & -2 \end{array}), & \text{if } i \text{ is odd} \\ \mathrm{Circ}(\underbrace{\begin{array}{ccc} 0 & \cdots & 0 \end{array}}_{m-n} \underbrace{\begin{array}{ccccc} -1 & 2 & -2 & \cdots & -2 \end{array}}_{n}), & \text{if } i \text{ is even.} \end{cases}$$

To find the nullity of the reduced crossing matrix M', it suffices to examine the lower right corner block R^T. It turns out that it is simpler to work with the transpose $R=(R^T)^T$ of this block. The form of this *residual matrix* R is as follows:

$$R = \mathrm{Circ}(\underbrace{1,-2,\cdots,2,-2,1}_{n+1},\underbrace{0,\cdots,0}_{m-n-1}). \tag{22}$$

Clearly the nullity of the crossing matrix M will be the nullity of the residual matrix R. We will begin our reduction of R by reducing the first $m-n$ rows. To do this we need to define an auxiliary matrix Γ. By the division algorithm we know that $m-n=qn+r$ for some $q,r\in\mathbb{Z}$ with $0\le r<n$. Define a

circulant matrix Γ whose initial row has q sections of size n, and a final section of size r, as follows:

$$\Gamma = \mathrm{Circ}\left(\underbrace{\underbrace{1 \;\; 2 \;\; \cdots \;\; 2 \;\; 2}_{n} \underbrace{3 \;\; 4 \;\; 4 \;\; \cdots \;\; 4 \;\; 4}_{n} \underbrace{5 \;\; 6 \;\; 6 \;\; \cdots \;\; 6 \;\; 6}_{n} \;\; \cdots}_{m-n} \right).$$

The i^{th} row of Γ will be denoted by $(\gamma_{i,1}, \gamma_{i,2}, \cdots, \gamma_{i,m-n})$. We can now use the entries of Γ to reduce the first $m-n$ rows of R, as follows:

$$R_i \to \sum_{k=i}^{m-n} \gamma_{ik} R_k.$$

The sequence of row operations above result in a partially reduced matrix R' with an identity block in its upper left corner of size $m-n$. We can use this identity block to reduce the first $m-n$ entries of each of the remaining rows, as follows. Consider the $n \times n$ circulant matrix

$$\Delta = \mathrm{Circ}(-1, 2, -2, \cdots, 2).$$

We will denote the i^{th} row of Δ by $(\delta_{i,1}, \delta_{i,2}, \cdots, \delta_{i,n})$. From the form of R shown in Equation 22 it is clear that we can reduce the bottom left corner of R' by the following sequence of row operations:

$$R_{m-n+j} = \begin{cases} R_{m-n+j} + 2(\sum_{k=1}^{j} \delta_{kj} R_k) & \text{if } j \leq m-n \\ R_{m-n+j} + 2(\sum_{k=1}^{m-n} \delta_{kj} R_k) & \text{if } j > m-n \end{cases} \tag{23}$$

We now have a further reduced matrix R''. It remains only to reduce the entries of R'' on the right hand side of the block. this rectangle is an $n \times m$ block. This can be done using the following sequence of row operations: for $m-n+1 \leq j \leq m-2$, perform the operation in Equation (24) to row R_j and then the operation in Equation (25) to every row R_i with $1 \leq i \leq j-1$ and with $j+1 \leq i \leq m$.

$$R_j = \begin{cases} R_j - R_{j+2} & \text{if } j \text{ is even} \\ R_j + R_{j+1} & \text{if } j \text{ is odd} \end{cases} \tag{24}$$

$$R_i = R_i - r_{ij} R_j \tag{25}$$

The row operations above leave us with the following upper-trianglular matrix R''':

$$R''' = \begin{pmatrix} 1 & & & & & -(m-1) & m-2 \\ & 1 & & & & -(m-2) & m-3 \\ & & \ddots & & & \vdots & \vdots \\ & & & 1 & & -3 & 2 \\ & & & & 1 & -2 & 1 \\ & & & & & m & m \\ & & & & & 0 & 0 \end{pmatrix}$$

By assumption we know that $T_{m,n}$ is p-colorable and m is odd, and therefore by Main Theorem 1 we know that p divides m. Therefore the matrix R''' clearly has p-nullity 2, and thus the torus knot $T_{m,n}$ must have p-nullity 2. $\square$

5 Further Questions

(1) We noticed in Section 4.1 that the main coloring of $T_{m,n}$ not only uses all of the colors but each color is used the same number of times as all other colors. What is the significance of this color distribution and what other knots have a similar color distribution?

(2) The idea of a variance vector was also introduced in Section 3.2 and used extensively for the geometric proof of both Main Theorems 1 and 2. Can this idea be applied other types of knots, especially those whose braid word is some power of a base word?

(3) Our reduction of M in Section 4.2 is applied to a specific case of $T_{m,n}$. We found that Equation (21) can actually be applied to any knot $T_{m,n}$ where without loss of generality m is odd and n is even. We found that if $m > n$ and m is even then

$$R = \mathrm{Circ}(\underbrace{1, -2, \cdots, 2, -1}_{n+1}, \underbrace{0, \cdots, 0}_{m-n-1}).$$

We then found that although the reduction process of R seems similar to the one shown in Section 4.2, the process breaks down into an unwieldy number of cases. We would be interested in finding a generalized reduction for this matrix.

(4) How can we apply the concept of p-coloring classes to other types of knots in order to categorize their p-colorability?

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