COLORING INVARIANT AND DETERMINANTS

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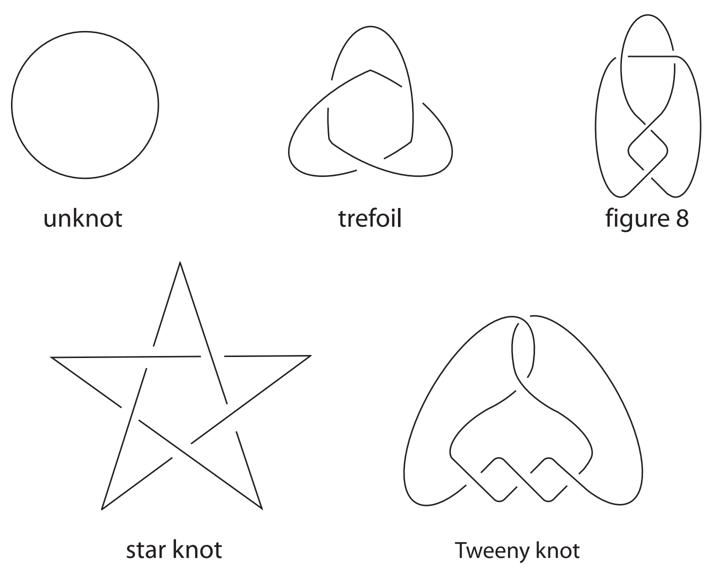
ABSTRACT. For a few weeks, I have studied the coloring invariant and the matrices associated with it. I have studied different ways to find the determinant of a pretzel knot and have focused on a few conjectures: 1) The determinant of a (n, m) pretzel is $|n+m|$. 2) A (n, m) pretzel is a link when both m and n are even or odd. Otherwise the pretzel is a knot. 3) The determinant is divisible by 2 iff the pretzel is a link.

1. INTRODUCTION: WHAT IS A KNOT?

What is a knot? A knot can be thought of, simply as a loop of rope with no end and no beginning, like tying your shoe strings together and then gluing the ends to one another. Adams describes a knot as a closed curve in space that does not intersect itself anywhere. ([Adams, p2]) Some of the famous knots are:



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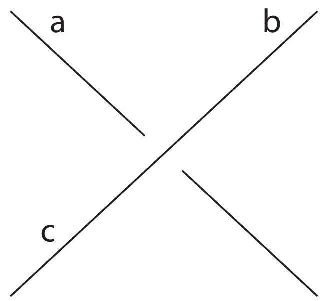
2. Coloring Invariant

One way to distinguish certain knots is through the coloring invariant. A knot or link is n-colorable, where n is prime, provided:

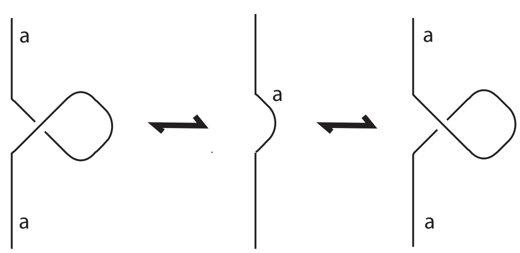
a. Its arcs can be labeled with colors such that $2a \equiv b + c (\bmod n)$ at each crossing, where a, b, c are defined by:



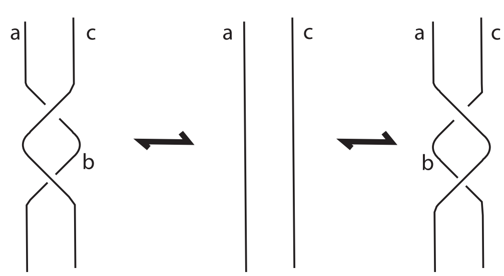
b. There are at least two colors present in the coloring.

Theorem 1. *If K is a knot or link that can be colored* mod$n$*, then every projection of K can be colored* $(\bmod\ n)$.

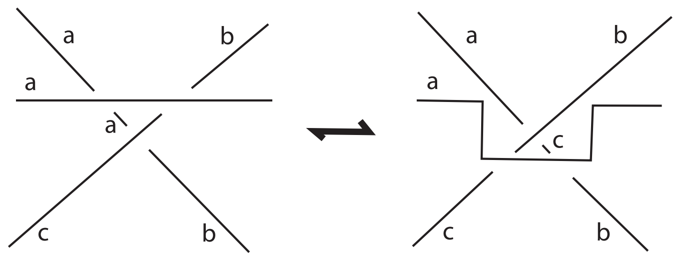
Proof: Check the Reidemeister moves



R1: Every arc involved remains the same color.



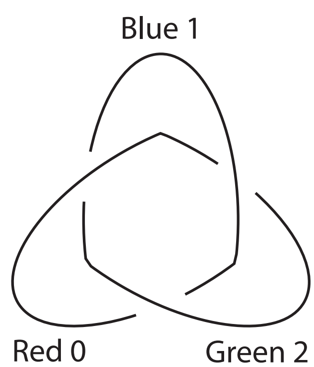
R2: An arc is added or removed. Although, we may gain a color or lose a color, there will still be at least two colors present.



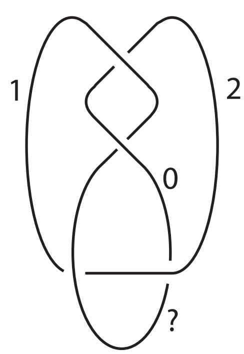
R3: (Example) Parts of the arcs will change color, but all of the crossings will still follow the criteria since the ends never change colors.

□

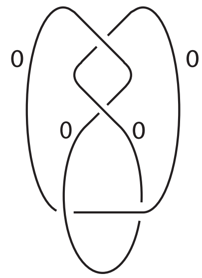
The trefoil is an example of a mod 3-colorable knot.



Next, try to color the figure-8 knot using the three colors.

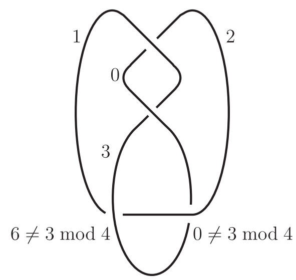


No way to color the last arc without violating the first rule.

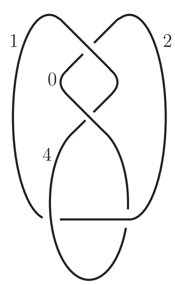


This coloring violates the second rule.

Using the rules will help to check if other coloring systems will work.



The mod 4-coloring system violates the first rule.



The figure-8 knot is mod 5-colorable.

Does the coloring system have something to do with the crossing number? That will be explored at a later point. First, we must introduce the idea of the determinant of a knot and how it relates to the coloring invariant.

3. Determinants

There is a matrix associated with each knot or link. The determinant of this matrix will tell you what coloring system(s) will work for the knot or link.

3.1. **Coloring Invariant Matrix.** The matrix of the coloring invariant consists of the equations formed at each crossing by the first rule, $2a \equiv b + c \bmod n \Rightarrow 2a - b - c = 0 \bmod n$. For example, the matrix for the figure-8 looks like this:

$$\begin{pmatrix} 1 & 2 & 0 & -1 \\ 0 & -1 & -1 & 2 \\ -1 & -1 & 2 & 0 \\ 2 & 0 & -1 & -1 \end{pmatrix}$$

$$\text{Determinant} = 0$$

Notice that the rows and columns of this matrix are not linearly independent. This corresponds to there being a trivial solution namely, the solution of coloring the whole knot or link one color. The second rule makes this solution invalid. To find non-trivial colorings, we must eliminate one row and one column and find the determinant of the remaining matrix.

$$\begin{pmatrix} -1 & 2 & 0 & \cancel{-1} \\ 0 & -1 & -1 & \cancel{2} \\ -1 & -1 & 2 & \cancel{0} \\ \cancel{2} & \cancel{0} & \cancel{-1} & \cancel{-1} \end{pmatrix} \Rightarrow \begin{pmatrix} -1 & 2 & 0 \\ 0 & -1 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

$$\text{Determinant} = 5$$

This determinant means that the figure-8 knot can be colored mod 5. Another property of the determinant is that it tells you all of the coloring systems you can use.

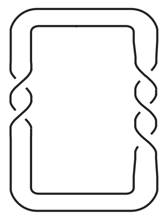
Lemma 2. *A knot is n-colorable iff $n \mid d$, where d is the determinant of the knot.*

Proof: ($\Rightarrow$) Assume K is n-colorable. WTS $n \mid d$. Since K is n-colorable, there exist a $\mathbf{v}$ such that $A\mathbf{v} \equiv \mathbf{0} \bmod n$ and $\mathbf{v} \not\equiv 0 \bmod n$. This means that the determinant d of A is $0 \bmod n$ or $n \mid d$.

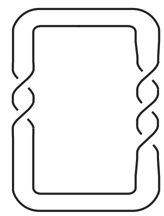
($\Leftarrow$) Assume $n \mid d$. WTS K is n-colorable. Since $n|d$, we know $d \equiv 0 \bmod n$. This means that there is a non-zero vector, $\mathbf{v}$, such that $A\mathbf{v} \equiv 0 \bmod n$. □

4. Pretzel Links

Pretzel links are tangles represented by a finite number of integers that represent the number of crossings in each tower. Alternating pretzels are pretzels whose integers all have the same sign. The sign of the integers denotes a positive or negative slope of the overcrossing. Non-Alternating pretzels have integers which are not all the same sign.



$(2,-3)$ Pretzel

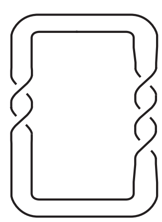


$(2,3)$ Pretzel

We will look at two different conjectures about (n,m) pretzel links. (See [DMU].)

A) The determinant of a (n,m) pretzel is $|n+m|$.

First we will look at a simple pretzel link:



$(2,3)$ Pretzel

Color invariant matrix

$$\begin{pmatrix} -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & 0 & -1 & 2 \\ 2 & -1 & 0 & 0 & -1 \end{pmatrix}$$

After eliminating one row and one column, we get:

$$\begin{pmatrix} -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & 0 & -1 & 2 \\ 2 & -1 & 0 & 0 & -1 \end{pmatrix} \Rightarrow \begin{pmatrix} -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \\ -1 & 0 & 0 & -1 \end{pmatrix}$$

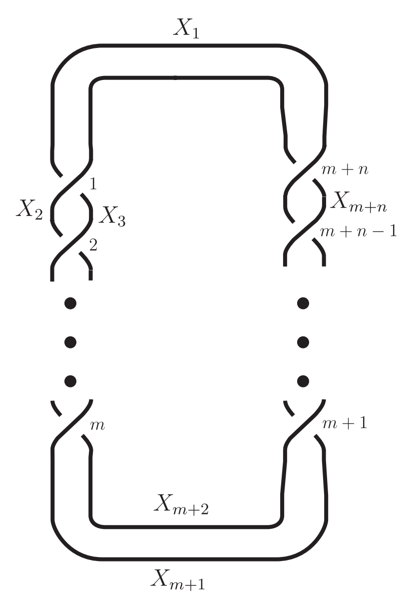
$$\text{Determinant } = 5$$

The coloring invariant shows that this conjecture is correct for this example. Now we will look at the general case.

Theorem 3. *The determinant of a (m, n) pretzel is $|m + n|$.*

Proof: We can interchange m and n by rotating the link so there are 4 cases: 1) $m, n > 0$; 2) $m, n < 0$; 3) $m = 0$; and, 4) $m < 0$ and $n > 0$.

Case 1) (Proof by Dass, McGrath, and Urbanski [DMU]) This case depends on a specific labeling of the link. First you must start with the upper left crossing and label that 1. Then going counterclockwise, label each consecutive crossing 2, then 3, then 4 and so on. For the arcs, label the uppermost arc X_1. Then label all the other arcs such that the over crossing at crossing m is X_{m+1}. The following $(m+n)$ by $(m+n)$ matrix will result from that labeling:



$$\begin{pmatrix} -1 & 2 & -1 & 0 & \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & -1 & 2 & -1 & 0 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot & 0 & -1 & 2 & -1 \\ -1 & 0 & \cdot & \cdot & \cdot & \cdot & 0 & -1 & 2 \\ 2 & -1 & 0 & \cdot & \cdot & \cdot & \cdot & 0 & -1 \end{pmatrix}$$

After eliminating the last row and the first column, we are left with this $(m+n-1)$ by $(m+n-1)$ matrix:

$$\begin{pmatrix} -1 & 2 & -1 & 0 & \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & -1 & 2 & -1 & 0 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot & 0 & -1 & 2 & -1 \\ -1 & 0 & \cdot & \cdot & \cdot & \cdot & 0 & -1 & 2 \\ 2 & -1 & 0 & \cdot & \cdot & \cdot & \cdot & 0 & -1 \end{pmatrix} \Rightarrow$$

$$\begin{pmatrix} 2 & -1 & 0 & \cdot & \cdot & \cdot & \cdot & 0 \\ -1 & 2 & -1 & 0 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 0 & -1 & 2 & -1 \\ 0 & \cdot & \cdot & \cdot & \cdot & 0 & -1 & 2 \end{pmatrix} \Rightarrow$$

Then, after exchanging rows, we get a matrix that looks like this:

$$\begin{pmatrix} -1 & 2 & -1 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & -1 & 2 & -1 & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & -1 & 2 & -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & \cdot & -1 & 2 \\ 2 & -1 & 0 & 0 & \cdot & \cdot & \cdot & 0 \end{pmatrix} \Rightarrow$$

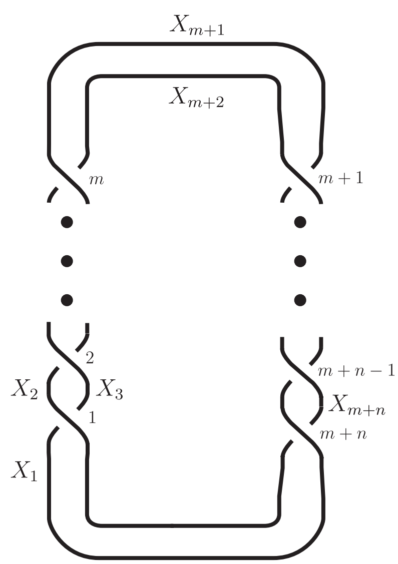
Notice that the last row is not in row reduced form. Next we will perform row operations such that R_{m+n-1} is replaced with $iR_{i-1} +$

R_{m+n-1}. For example, the first operation will replace R_{m+n-1} with $2R_1 + R_{m+n-1}$. This results in this $(m+n-1)$ by $(m+n-1)$ matrix:

$$\begin{pmatrix} -1 & 2 & -1 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & -1 & 2 & -1 & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & -1 & 2 & -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & \cdot & -1 & 2 \\ 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & m+n \end{pmatrix}$$

The determinant of a reduced matrix is the product of the diagonal entries. So the link determinant is $|(-1)^{(m+n-2)}(m+n)| = |m+n|$.

Case 2) This case also uses a specific labeling of the pretzel. First, label the lower left crossing 1. Then going clockwise, label each consecutive crossing 2, 3, and so on. The lowest arc will be labeled X_1 and all the other arcs will be labeled such that the over crossing at crossing m is X_{m+1}. Using this label system, the proof will follow exactly like the proof for case 1.

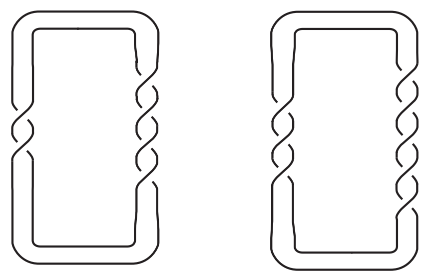


Case 3) There are 3 possibilities, $n > 0$, $n < 0$ and $n = 0$. When $n > 0$, the case 1 labeling and proof will prove this step. When $n < 0$, the proof and labeling of case 2 will show that this is true. When $n = 0$ we have the unlink. We know that the unlink has determinant of $0 = |m+n|$, as needed.

Case 4) Notice that this pretzel can be reduced to an $(m+n,0)$ pretzel link. We know from case 3 that the determinant of this pretzel will be $|m+n|$. $\square$

B) A (m,n) pretzel is a link when both m and n are even or odd. Otherwise the pretzel is a knot.

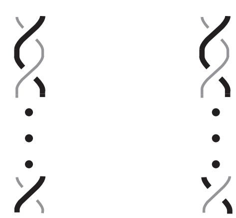
By examining a few examples, this appears to be true.



$(2,4)$ $(3,5)$

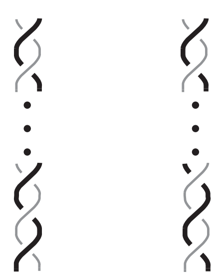
Lemma 4. *Adding or subtracting two crossings from a tangle (i.e. tower) in a pretzel will not change it from knot to link or vice versa.*

Proof: First, we will start with towers with n-crossings.



n is odd n is even

If we add an even number of crossings to this projection we get:



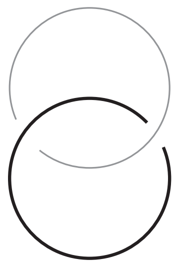
Notice, that no matter what we started with, since the strands end in the same pattern as before, we end up with the same thing we started with, knot or link. $\square$

Theorem 5. *A (m, n) pretzel is a link when both m and n are even or odd. Otherwise the pretzel is a knot.*

Proof:

Case 1 (n, m both odd) We will use Induction

Base: $n, m = 1$

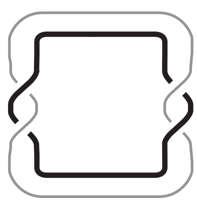


is a link.

Induction: Assume (n, m) is a link with n and m both odd. WTS $(n+2, m)$ and $(n, m+2)$ are links. By the above Lemma, we know that these are still links, as needed. $\square$

Case 2 (n, m both even) We will use Induction

Base: $n, m = 2$

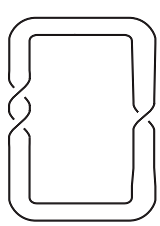


Yes, it is a link.

Induction: Assume (n, m) is a link where n and m are both even. WTS $(n+2, m)$ and $(n, m+2)$ are links. By the above Lemma, we know that these are still links, as needed.

Case 3 (n is even and m is odd) We will use Induction

Base: $n = 2$, $m = 1$



Yes, this is a knot

Induction: Assume (n, m) is a knot where n is even and m is odd. WTS $(n+2, m)$ and $(n, m+2)$ are knots. By the above Lemma, we know that these are still knots, as needed.

Case 4 (n is odd and m is even)

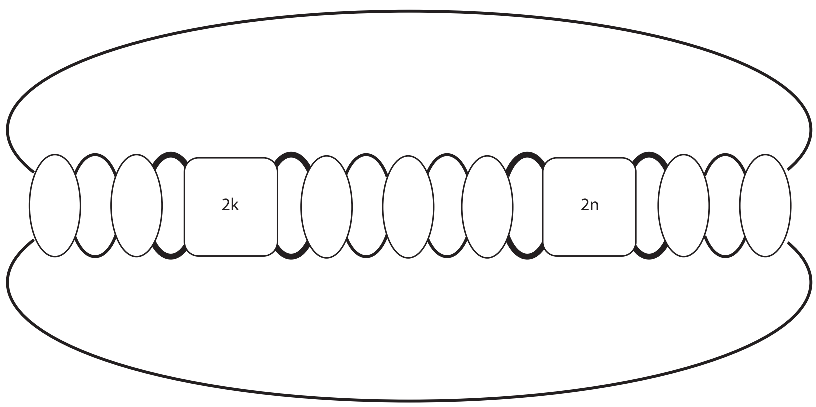
Since we can interchange m and n by rotating the pretzel, by Case 3, this is a knot. □

This proof works for pretzels with two towers, but how do we know we have a link generally?

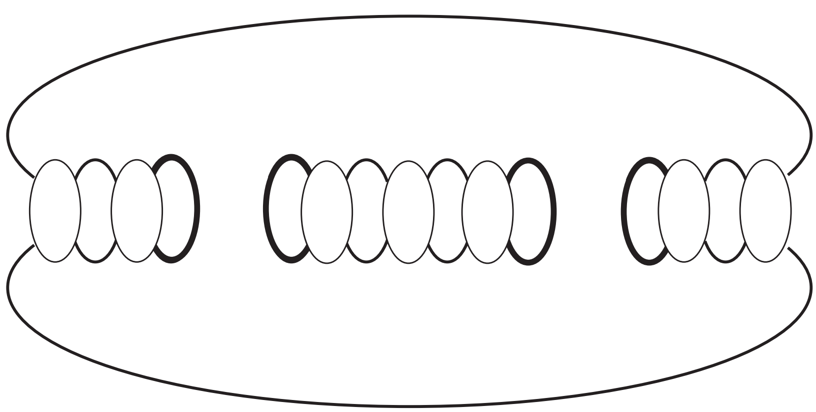
Lemma 6. *A pretzel link is a link (and not a knot) if there are at least two even towers or an even number of odd towers with no even towers. Otherwise, it is a knot.*

Proof:

Case 1: If a pretzel has 2 even towers:

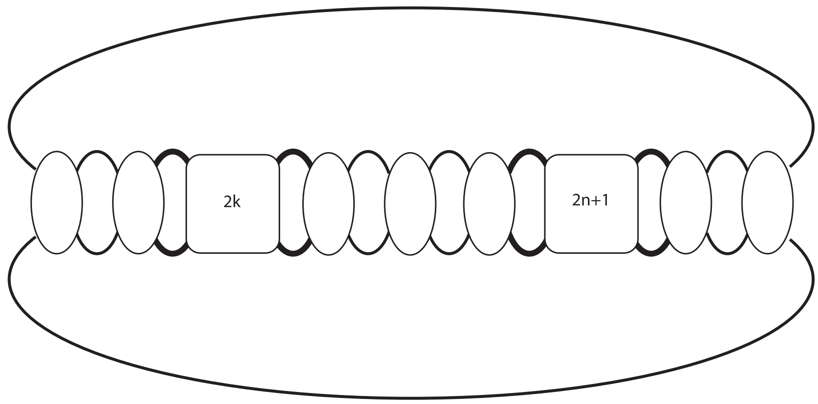


Since adding or subtracting 2 crossings doesnt change the fact that the pretzel is a link or a knot, let $k, n = 0$ (see Lemma above). We then have:

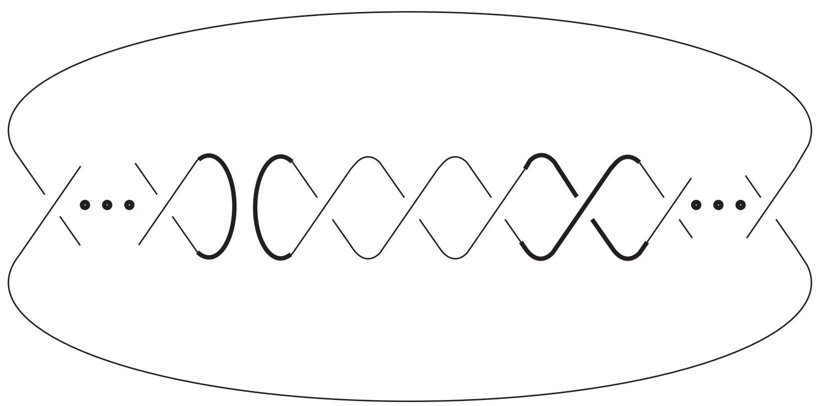


a link. Notice that this will work for any pretzel with 2 or more even towers, because there will always be at least two towers to do this with.

Case 2: If a pretzel has exactly 1 even tower:

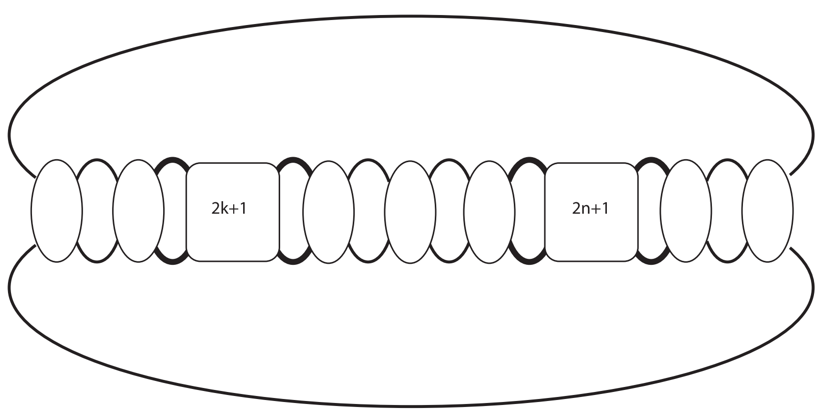


Using Lemma, let k, n=0. We now have:

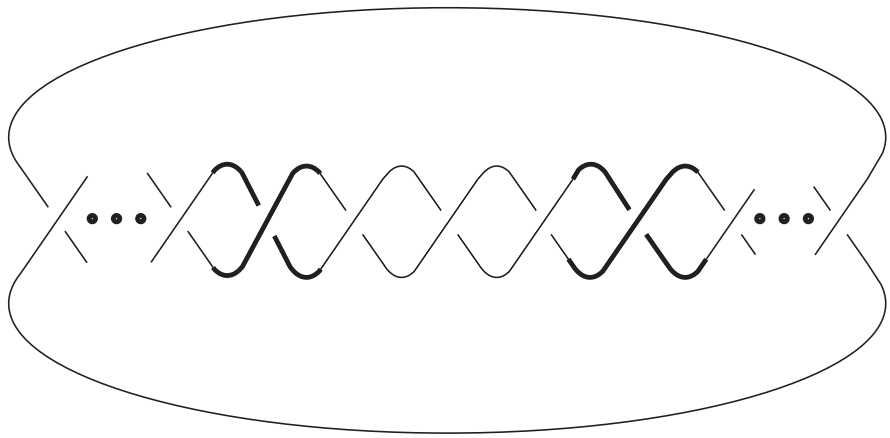


A knot.

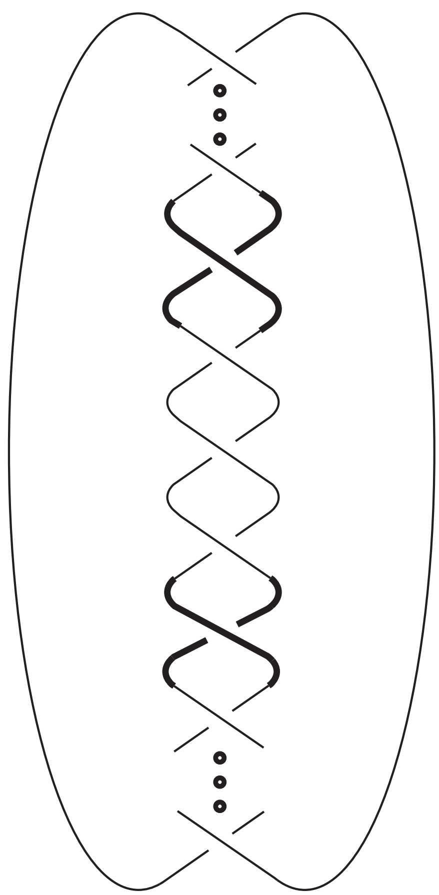
Case 3: If a pretzel has no even towers:



Let $k, n = 0$. We now have:



By rotating the projection 90 degrees we get an $(m, 0)$ pretzel where m is the number of original towers:



We know from the previous proof that if m is even then the pretzel is a link. If m is odd, then we have a knot. □

Next, I looked at whether looking at the determinant helped with figuring out whether or not the pretzel is a link.

Theorem 7. *A Pretzel is a link iff the determinant is divisible by two.*

Proof:

($\Rightarrow$) Assume the pretzel is a link. WTS that the determinant is divisible by 2. We know that the ways to get a pretzel link is by either having two or more even towers, or no even towers with an even number of odd towers. The formula for the determinant of a pretzel knot is $\sum p_1 p_2 \ldots p_{i-1} p_{i+1} \ldots p_n$. Notice that this number will be even if two towers are even since you are summing even numbers. Notice this number will also be even if there is an even number of odd towers with no even towers since you are taking the sum of an even number of odd numbers.

($\Leftarrow$) Assume two divides the determinant. WTS that the pretzel is a link. We know that the determinant of a pretzel is $\sum p_1 p_2 \ldots p_{i-1} p_{i+1} \ldots p_n$. For this number to be even, we need:

- At least two p_n to be even, or
- An even number of odd p_n with no even p_n.

If we have either of these criteria, then we have a link.

5. Further Exploration

For further exploration, I will be looking into a labeling system for non-alternating $(m_1, m_2, \ldots, m_n)$ pretzels to find a way to calculate the determinant. Other investigations will include the Kauffman conjecture which states that if the determinant of an alternating knot is prime, then no colors will be repeated in any non-trivial coloring of the knot.

References

[Adams] C. Adams, The Knot book, W.H. Freeman and Company, New York, 1994

[DMU] N. Dass, J. McGrath, E. Urbanski, The Determinant of a (m, n) Pretzel, Π*ME Journal*, Vol.11, No.3, pp135-137, 2000

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