1.6 Factoring

1.6.1 Overview

Factoring provides a powerful algebraic tools to analyze where an expression equals zero. In the previous section, we noted that a product of two factors can only equal zero if at least one of the factors itself equals zero (see Theorem 1.5.12). More generally, factoring helps us understand the relation between a product and zero, whether it is greater than or less than zero.

In this section, we will review some strategies for factoring expressions. We introduce some technological approaches to factoring. Then we review some strategies that you would have seen in an algebra course. We also use our knowledge of how factoring to develop meaningful models.

1.6.2 Factoring with Technology

Factoring is a process that is best accomplished using technology. Every technique that we learn by hand can be accomplished much more quickly and more reliably by a computer. Once the computer has been programmed correctly, it doesn't commit the arithmetic errors that we are prone to make. In a practical setting, except for simple problems, you will be better off obtaining factors from a computer algebra system.

A computer algebra system (CAS) is a computer program that is designed to apply the rules of algebra according to the user's request. The popular website WolframAlpha (`https://www.wolframalpha.com`) allows you to ask mathematical questions using natural language. It interprets your request and shows a variety of mathematical responses that might answer your question. For example, to factor a polynomial like $x^3 - 7x + 6$, we would submit a request `factor x^3-7x+6`. WolframAlpha would give a response that the result is $(x-1)(x-2)(x+3)$.

WolframAlpha can perform many other basic computations. It is built on the same CAS as a stand alone application called Mathematica, which is sold by Wolfram. A web-based system like WolframAlpha has a disadvantage that we can not create a chain of dependent calculation. Systems like Mathematica that use scripts or notebooks, on the other hand, do allow for interrelated calculations. Popular commercially available CAS programs include Wolfram's Mathematica and MapleSoft's Maple. Many college campuses have license agreements with one of these programs.

A free and open-source alternative CAS is SageMath (`http://www.sagemath.org/`). While you can download and use this program on your own computer, you can also access and use its capabilities through web-access. Similar to using WolframAlpha to use the power of Mathematica, you can use the power of SageMath in what are known as SageCells. A SageCell can be accessed at `https://sagecell.sagemath.org`. The online version of this text also has live SageCells embedded as interactive demonstrations.

In SageMath, we can create mathematical objects (like expressions or equations) and then perform actions on those objects. The creation and naming of a mathematical object occurs through an *assignment*. For example, to create the expression $x^3 - 7x + 6$ and assign it to a name `expr1`, we type the command `expr1 = x^3-7*x+6` on its own line. Notice how we must explicity state that there is multiplication between 7 and x. To create a new expression that is the factored form and name it `expr2`, we perform a new assignment where the value is based on the factoring action applied to `expr1`. The relevant command would be `expr2 = expr1.factor()`.

The SageCell script below illustrates the commands working together. There are also two commands that show us a nicely formatted version of our expressions for comparison. Be sure to try this script. Push the Evaluate button in the live online cell, or copy this into a clean SageCell and evaluate it there. You should see the results:

$$x^3 - 7x + 6$$

$$(x + 3)(x - 1)(x - 2)$$

```
expr1 = x^3-7*x+6
expr2 = expr1.factor()
show(expr1)
show(expr2)
```

Using named expressions is useful when we have additional actions to do later. For simple problems like this, we can actually skip naming the expressions. Try changing the script to the following command and re-evaluate: show(factor(x^3-7*x+6)). What happens if you don't include the command show?

The reverse process of multiplying out a factored expression also is frequently needed. It is also tedious to do by hand and more reliable using a computer. The relevant command is to expand the expression. Suppose we want to know what $(x+1)(x+2)(x+3)(x+4)$ is as a polynomial in standard form. Using the SageCell script below reveals the answer to be

$$(x + 1)(x + 2)(x + 3)(x + 4) = x^4 + 10x^3 + 35x^2 + 50x + 24.$$

Again, notice how the CAS requires that we show explicitly where each multiplication occurs. What happens if there aren't parentheses? What do you think might be happening?

```
expr1 = (x+1)*(x+2)*(x+3)*(x+4)
expr2 = expr1.expand()
show(expr1)
show(expr2)
```

1.6.3 Strategies for Factoring by Hand

Although technology makes factoring fast and simple, we should be prepared to perform simple factoring by hand. We review some basic strategies for factoring that you would have learned in an algebra class.

It helps to remember that factoring is the reverse process of the distributive property of multiplication over addition. That is, when we expand $a(b+c) = ab + ac$, we see that ab and ac have the common factor of a from distribution. If we can identify a common factor, then we can reverse the process and write our expression as a multiplication over addition of terms.

Example 1.6.1 Factor $4x^2 + 6x^3$.

Solution.   Recalling that $4 = 2 \cdot 2$ and $6 = 2 \cdot 3$, we recognize that the terms $4x^2$ and $6x^3$ have a common factor of $2x^2$:

$$4x^2 + 6x^3 = 2x^2 \cdot 2 + 2x^2 \cdot 3x.$$

Factoring this out, we have

$$4x^2 + 6x^3 = 2x^2(2 + 3x).$$

□

More advanced factoring approaches often are built on top of this idea. For example, a method known as **factoring by grouping** arises by expressing a sum of terms as a sum of two groups of terms that can be found to have a common factor. Some cubic polynomials (but not most) can be factored using this approach.

Example 1.6.2 Use factoring by grouping to factor $x^3 - 3x^2 - 4x + 12$.

Solution. The strategy is to group the x^3 and x^2 terms together and to group the x and constant terms together,

$$x^3 - 3x^2 - 4x + 12 = (x^3 - 3x^2) + (-4x + 12),$$

and then factor out common factors. The first group $x^3 - 3x^2$ has a common factor of x^2 to give

$$x^3 - 3x^2 = x^2(x - 3).$$

The second group $-4x + 12$ has a common factor of 4 to give

$$-4x + 12 = 4(-x + 3).$$

To get a common factor, we should recognize that we should have used a common factor of -4:

$$-4x + 12 = -4(x - 3).$$

We now have groups with a common factor:

$$\begin{aligned} x^3 - 3x^2 - 4x + 12 &= (x^3 - 3x^2) + (-4x + 12) \\ &= x^2(x - 3) + -4(x - 3) \\ &= (x^2 - 4)(x - 3) \end{aligned}$$

A full factorization would also factor $x^2 - 4 = (x + 2)(x - 2)$ to give

$$x^3 - 3x^2 - 4x + 12 = (x + 2)(x - 2)(x - 3).$$

□

In many cases, a mathematical solution to a problem is easier to find when we anticipate what it should look like. We use this concept to guide us in factoring quadratic polynomials. Quadratic polynomials result from expanding a product of the form $(ax + b)(cx + d)$. That expansion is often described using the acronym *FOIL* (First-Outside-Inside-Last):

$$(ax + b)(cx + d) = ac\,x^2 + ad\,x + bc\,x + bd = ac\,x^2 + (ad + bc)x + bd.$$

Notice that in the middle expression, we have four terms, similar to what we had with cubic polynomials. This means that we might be able to factor if we can find a clever way to do grouping.

If we want to factor a quadratic expression $Ax^2 + Bx + C$, then we are looking for values for a, b, c, d so that

$$Ax^2 + Bx + C = (ax + b)(cx + d).$$

This requires that $A = ac$, $C = bd$, and $B = ad + bc$. A clever observation is that $AC = (ac)(bd) = (ad)(bc)$, so that we are writing B as a sum of factors of AC. This will be how we create our grouping.

Example 1.6.3 Factor $2x^2 - x - 6$.

Solution. We begin by recognizing the coefficients $A = 2$, $B = -1$, and

$C = -6$. We want to write $B = -1$ as a sum of factors of $AC = -12$. Our strategy is to think through all of the simple factors of -12 and see if any pair of factors add to -1. What are the factors of -12?

$$-12 = (-1)(12) = (1)(-12) = (-2)(6) = (2)(-6) = (-3)(4) = (3)(-4)$$

When we check the sum of those pairs, we find:

$$-1 + 12 = 11$$
$$1 + -12 = -11$$
$$-2 + 6 = 4$$
$$2 + -6 = -4$$
$$-3 + 4 = 1$$
$$3 + -4 = -1$$

In practice, we would likely add the factor pairs in our head rather than write them down.

Once we find the pair of factors with the correct sum, $3 + -4 = -1$, we expand the term $-x$ as a sum $3x - 4x$ to rewrite the quadratic as a sum of four terms that can now be grouped.

$$\begin{aligned}
2x^2 - x - 6 &= 2x^2 + 3x - 4x - 6 \\
&= (2x^2 + 3x) + (-4x - 6) \\
&= x(2x + 3) + -2(2x + 3) \\
&= (x - 2)(2x + 3)
\end{aligned}$$

We thus have the factors

$$2x^2 - x - 6 = (x - 2)(2x + 3).$$

$\square$

When using the method of grouping for quadratics, be sure that you consider both positive and negative factors as pairs.

Another example of anticipating the form of a solution occurs when we know a polynomial's root. Knowing that a polynomial has a root means that we also know a factor.

Theorem 1.6.4 Root–Factor Theorem. *Suppose $p(x)$ is a polynomial of degree n for which $x = c$ is a root, $p(c) = 0$. Then $p(x)$ can be written in a factored form*

$$p(x) = (x - c) \cdot q(x)$$

where $q(x)$ is a polynomial of degree $n - 1$.

If we can find a root to a polynomial, then we know a simple factor. One way to find the root is by looking at its graph. Knowing the original polynomial and a factor, we can work out the other polynomial factor. Finding that other factor corresponds to polynomial division.

Example 1.6.5 Factor the polynomial $6x^3 - 5x^2 - 17x + 6$.

Solution.   When we graph the polynomial, we can identify possible roots.
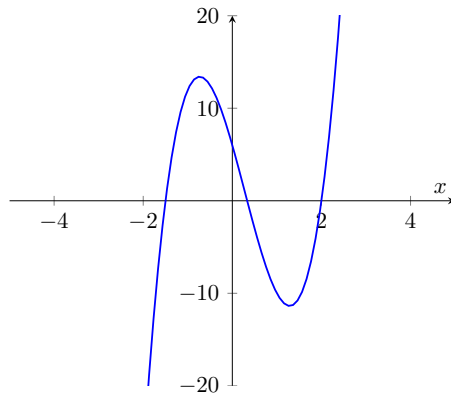
The graph suggests that there might be a root at $x = 2$, between -1 and -2, and between 0 and 1. We verify that $x = 2$ is a root by substituting that value into the formula:

$$6x^3 - 5x^2 - 17x + 6 = 6(2)^3 - 5(2)^2 - 17(2) + 6$$
$$= 6(8) - 5(4) - 17(2) + 6$$
$$= 48 - 20 - 34 + 6$$
$$= 0$$

The factor that corresponds to $x = 2$ as a factor is $x - 2$ (because $x = 2$ is equivalent to $x - 2 = 0$). We now know that there is a polynomial $q(x) = ax^2 + bx + c$ with degree 2 so that

$$6x^3 - 5x^2 - 17x + 6 = (x - 2) \cdot (ax^2 + bx + c).$$

We find $q(x) = ax^2 + bx + c$ by polynomial division. In effect, however, we are multiplying by this polynomial with unknown coefficients and determining the coefficient values so that the product equals the original polynomial.

$$(x - 2) \cdot (ax^2 + bx + c) = x \cdot (ax^2 + bx + c) - 2(ax^2 + bx + c)$$
$$= ax^3 + bx^2 + cx - 2ax^2 - 2bx - 2c$$
$$= ax^3 + (b - 2a)x^2 + (c - 2b)x - 2c$$

Because this product must equal $6x^3 - 5x^2 - 17x + 6$, we have a system of equations based on matching coefficients:

$$\begin{cases} a = 6 \\ b - 2a = -5 \\ c - 2b = -17 \\ -2c = 6 \end{cases} \qquad \Leftrightarrow \qquad \begin{cases} a = 6 \\ b = 2a - 5 \\ c = 2b - 17 \\ -2c = 6 \end{cases}$$

Substituting $a = 6$ into the second equation gives $b = 2(6) - 5 = 7$. Substituting $b = 7$ into the third equation gives $c = 2(7) - 17 = -3$. This matches the fourth equation solved for c. We can therefore write the factorization using the values of the coefficients for $q(x)$.

$$6x^3 - 5x^2 - 17x + 6 = (x - 2)(6x^2 + 7x - 3).$$

We finish the problem by factoring the new quadratic factor. The product $ac = 6(-3) = -18$ has factors $-18 = (-2)(9)$ that sum to $-2 + 9 = 7$. We can rewrite and group the quadratic to have common factors:

$$6x^2 + 7x - 3 = 6x^2 + -2x + 9x - 3$$

$$= (6x^2 - 2x) + (9x - 3)$$
$$= 2x(3x - 1) + 3(3x - 1)$$
$$= (2x + 3)(3x - 1)$$

Consequently, our final factorization can be written

$$6x^3 - 5x^2 - 17x + 6 = (x - 2)(2x + 3)(3x - 1)$$

The other roots can be found from the factors: $x = -\frac{3}{2}$ (from $2x + 3 = 0$) and $x = \frac{1}{3}$ (from $3x - 1 = 0$). □

1.6.4 Factors for Polynomial Modeling

Polynomials have easily understood behavior based on their factors. The roots of the factors are exactly the roots of the polynomial. Furthermore, these roots are the only possible locations where the polynomial might change sign. It is easy to show that each simple factor changes sign exactly at its root by solving an inequality directly. Then, knowing the signs of each factor allows us to determine the sign of the polynomial as a whole, based on the following theorem.

Theorem 1.6.6 *The relation of product of expressions $w = u_1 \cdot u_2 \cdots u_n$ with zero is based the relations of the factors:*

- *The product $w = 0$ if and only if at least one $u_k = 0$.*

- *The product $w > 0$ if and only if all $u_k \neq 0$ and there are an even number of $u_k < 0$.*

- *The product $w < 0$ if and only if all $u_k \neq 0$ and there are an odd number of $u_k < 0$.*

We can analyze the behavior of a polynomial, in terms of its relation to zero, using the factors. This process is called **sign analysis**.

1. Find the factored version of the polynomial.

2. Identify the roots of all of the factors and order them on the number line.

3. The roots divide the number line into a collection of intervals. On each interval between roots, count the number of factors that will be negative.

4. On each interval, if the number of negative factors is even, then the polynomial will be greater than zero. If the number of negative factors is odd, then the polynomial will be less than zero.

Sometimes, a polynomial has repeated roots that appear as a factor raised to a power. When counting factors, the power is used as multiplicity of repetition.

Example 1.6.7 Perform sign analysis of the polynomial

$$p(x) = x^2(x + 2)(2x - 5).$$

Solution. The polynomial has four factors—x (double), $x + 2$, and $2x - 5$. The roots of these factors are $x = 0$, $x = -2$, and $x = \frac{5}{2}$. We order these roots graphically on a number line.
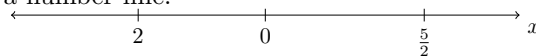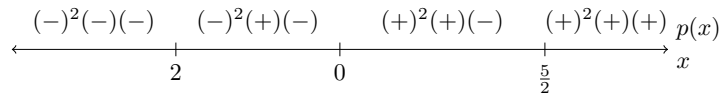

Once the roots are ordered on the number line, we can see the intervals of interest. Intervals represent continuous segments of the number line and are described as a range from left to right. There are four intervals: $(-\infty, -2)$,

$(-2,0)$, $(0,\frac{5}{2})$, and $(\frac{5}{2},\infty)$. On each interval, we can determine the sign of each factor, illustrated in the table below. Then, when we multiply these factors together, we find the overall sign of the polynomial. The polynomial is negative when there is an odd number of negative factors and positive when there is an even number of negative factors.

Interval	x^2	$x+2$	$2x-5$	$p(x)$
$(-\infty,-2)$	$(-)^2$	$-$	$-$	$+$
$(-2,0)$	$(-)^2$	$+$	$-$	$-$
$(0,\frac{5}{2})$	$(+)^2$	$+$	$-$	$-$
$(\frac{5}{2},\infty)$	$(+)^2$	$+$	$+$	$+$

Rather than create a table of signs, we can label the signs of the factors directly on the number line above each interval.

$$(-)^2(-)(-) \qquad (-)^2(+)(-) \qquad (+)^2(+)(-) \qquad (+)^2(+)(+) \; p(x)$$

$$\xleftarrow{\hspace{1cm}} \underset{2}{\vert} \qquad\qquad \underset{0}{\vert} \qquad\qquad \underset{\frac{5}{2}}{\vert} \xrightarrow{\hspace{1cm}} x$$

The table or the number allows us to interpret the relation of the polynomial with zero. The polynomial is positive (greater than zero) on the intervals $(-\infty,-2)$ and $(\frac{5}{2},\infty)$ and negative (less than zero) on the intervals $(-2,0)$ and $(0,\frac{5}{2})$. When we look at a graph of the polynomial, we can see that the graph is above the axis on the outer intervals and below the axis on the inner intervals.

□
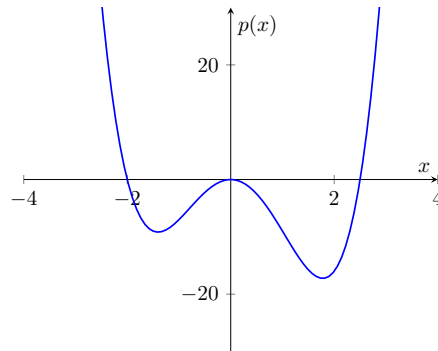
Performing sign analysis on a polynomial might seem like a silly exercise. After all, with our graphing calculators and computers, it is easy enough to graph the polynomial directly and look where the graph is above or below the axis. It is in the reverse process that we start to see the power.
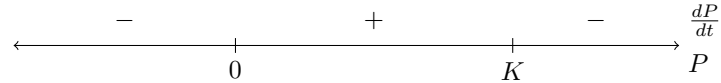
Suppose that we want to explore a mathematical model for a phenomenon where we know how a quantity relates with zero on the number line. However, we would like to have a simple mathematical formula that captures that relation. By constructing a polynomial with factors that will match the sign analysis, we can use that polymomial as our model.

Example 1.6.8 A simple model for density-dependent population growth has a population's growth rate as positive when the population is between zero and a carrying capacity and negative when the population is above the carrying capacity. Create a simple polynomial model that will capture this behavior.

Solution. We will let P be our symbol to represent the value of the population. In biology, the carrying capacity is most often symbolized by the symbol K. The symbol for a quantity's growth rate is called the **derivative** with respect to time and has the symbol $\frac{dP}{dt}$. (The goal of calculus is to understand what this derivative really means.) We want to create a model that describes

how the growth rate $\frac{dP}{dt}$ depends on the population P such that this formula is positive when P is in the interval $(0, K)$ and negative when P is in (K, ∞).

We can start by creating a number line summary of the behavior we want. Because $P < 0$ is not physically relevant, the sign used on $(-\infty, 0)$ doesn't matter. However, since polynomials change sign at roots unless a factor has an even power, we will choose to make $\frac{dP}{dt}$ negative when $P < 0$.

$$\xleftarrow{\quad\qquad\underset{\displaystyle 0}{-}\qquad\qquad\underset{\displaystyle}{+}\qquad\qquad\underset{\displaystyle K}{-}\qquad}\xrightarrow{} \begin{array}{c}\frac{dP}{dt}\\ P\end{array}$$

The roots help us know our basic factors. A root at $P = 0$ corresponds to a factor of P. A root at $P = K$ corresponds to a factor of $P - K$. The product $P(P - K)$, however, would have the opposite signs on the intervals. This is corrected by multiplying by a third constant factor, $-a$, where a itself is some positive constant. This gives us our basic model,

$$\frac{dP}{dt} = -aP(P - K).$$

If we multiply the negative sign by the factor $P - K$, we obtain an equivalent model

$$\frac{dP}{dt} = aP(K - P).$$

The constants a and K become model parameters.

In biology, slightly different model parameters are more commonly used. We rewrite our factor $K - P = K - \frac{PK}{K}$ and then factor out the common factor K We now have a model

$$\frac{dP}{dt} = aK\,P(1 - \frac{P}{K}).$$

The product aK can be replaced by another parameter, r, to obtain

$$\frac{dP}{dt} = r\,P(1 - \frac{P}{K}).$$

The parameter r is called the intrinsic per capita growth rate. This model is known as the **logistic growth rate model** for density-dependent growth. $\square$

In our model, we used a factor $1 - \frac{P}{K}$ instead of the factor $K - P$. These two factors have the same roots. One advantage to a factor like $1 - \frac{P}{K}$ is that the units of P and K cancel so that the factor is dimensionless. Polynomial models are often written with factors of the form $\frac{x}{a} - 1$ or $1 - \frac{x}{a}$ where $x = a$ is the desired root. In particular, a linear model with known x- and y-intercepts is most easily modeled using factors and does not require finding the slope.

Example 1.6.9 Suppose S is the number of seeds a plant produces and D is the density of competing plants around it. When there is no competition, $D = 0$, the plant can produce its highest output, $S = M$ seeds. When the competition reach a critical level, $D = D_M$, the plant no longer produces seed, $S = 0$. Develop a linear model for how S relates to D for $0 \le D \le D_M$.

Solution.   A linear model has a single root, which in this modeling scenario is $S = 0$ at $D = D_M$. This means we have a factor $D - D_M$ or $\dfrac{D}{D_M} - 1$. We need another constant factor, say A to account for the vertical scale, such as

the other intercept. Our model has the form

$$S = A\left(\frac{D}{D_M} - 1\right).$$

When we substitute $D = 0$, we find $S = -A = M$ so that the parameter $A = -M$. The model can then be written in terms of the data provided,

$$S = -M\left(\frac{D}{D_M} - 1\right) = M\left(1 - \frac{D}{D_M}\right).$$

$\square$
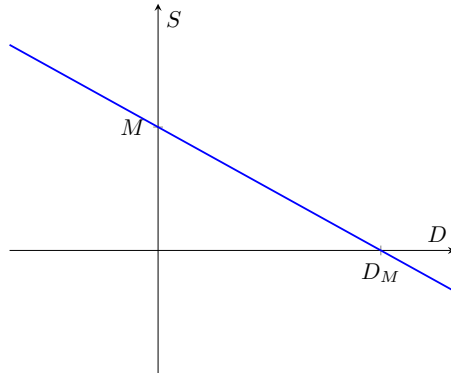
1.6.5 Summary

1. In practical applications, technology is the most efficient method to factor expressions. Computer Algebra Systems (CAS) like SageMath provide tools to perform operations on mathematical objects.

2. The first priority in factoring involves common factors. Grouping terms sometimes allows us to find common factors among the groups.

3. Quadratic polynomials with integer coefficients $Ax^2 + Bx + C$ can be factored by grouping if you can find factors of AC that add to the value of B. This is accomplished by rewriting the term Bx as two terms using the factors and then grouping terms.

4. Knowing a root $x = a$ of a polynomial tells us that $x - a$ is a factor. The other factor can be found by polynomial division.

5. The factored form of an expression allows us to perform sign analysis.

   (a) Identify the roots of all of the factors and order them on the number line.

   (b) The roots divide the number line into a collection of intervals. On each interval between roots, count the number of factors that will be negative.

   (c) On each interval, if the number of negative factors is even, then the expression will be greater than zero. If the number of negative factors is odd, then the expression will be less than zero.

6. Alternatively, knowing the roots and desired results for sign analysis, we can use a factored polynomial to generate a model for the relation between two variables.

1.6.6 Exercises

Use technology to factor the following formulas.

1. $x^3 + x^2 - 5x + 3$
2. $x^4 - 4x^3 - 11x^2 + 30x$
3. $x^4 + 10x^3 + 35x^2 + 50 + 24$
4. $24x^3 + 14x^2 - 11x - 6$

Use the method of grouping to factor each cubic.

5. $x^3 - 2x^2 + 3x - 6$
6. $x^3 - 5x^2 - 3x + 15$
7. $4x^3 - 12x^2 - x + 3$

Factor each quadratic polynomial.

8. $x^2 - 2x - 3$
9. $x^2 - 9x + 20$
10. $x^2 + 4x - 21$
11. $2x^2 + 3x - 2$
12. $2x^2 - 7x - 15$
13. $6x^2 + 11x - 10$

For each problem, verify that the given value is a root of the given polynomial. Use the Root-Factor theorem and polynomial division to factor the polynomial.

14. $x^3 - 6x^2 - x + 30$; $x = -2$
15. $x^3 - 2x^2 - 11x + 12$; $x = 4$
16. $2x^3 - x^2 - 13x - 6$; $x = 3$

For each factored polynomial, complete sign analysis to describe the intervals where the polynomial is greater than zero and where it less than zero. Then compare your results with a graph.

17. $3x(x - 4)$
18. $2x(2 - x)(3 - x)$
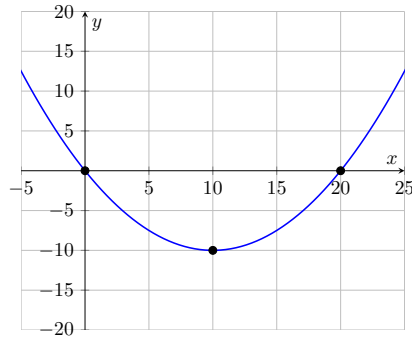19. $x(x + 2)(x - 1)^2$
20. $(x - 4)(2x - 3)(3x + 1)^3$

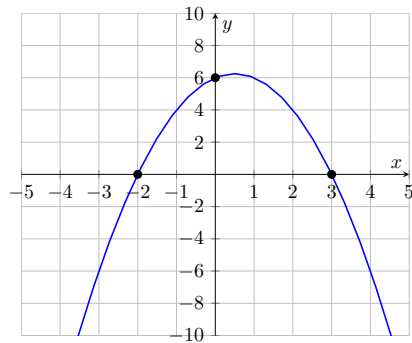**21.** Find a linear model for the following graph without finding the slope.
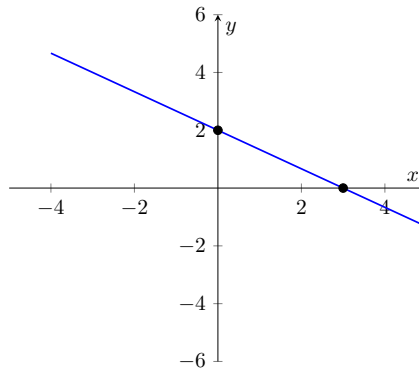
22. Find a linear model for the following graph without finding the slope.
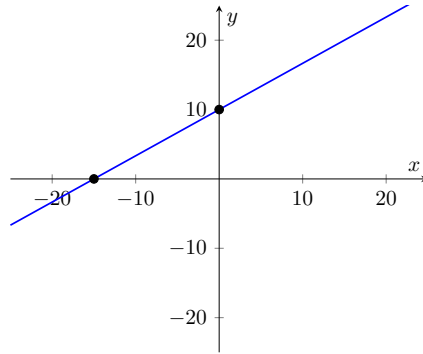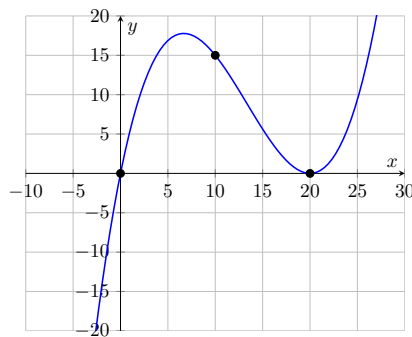

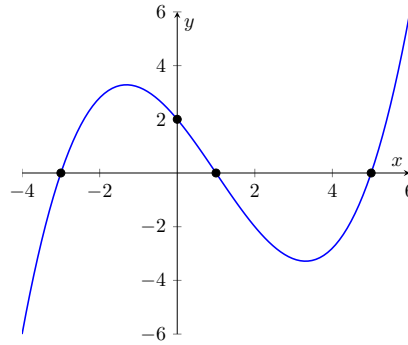**23.** Find a polynomial model for the following graph.

24. Find a polynomial model for the following graph.
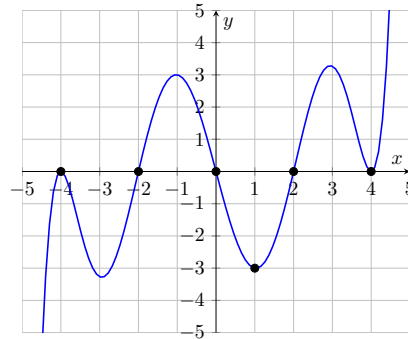

25. Find a polynomial model for the following graph.

26. Find a polynomial model for the following graph.

27. Find a polynomial model for the following graph.

28. In economics, the demand for a product, which measures the number of units a company can sell, is related to the price charged to purchase the product. Let p be the price being charged and let n be the demand or number of units that are sold. A simple model is that the demand has a linear relation with the price. Suppose that if the company gave away product for free, $p = 0$, the demand is $n = 5000$. On the other hand, if the price charged is $p = 50$, there is no demand, $n = 0$.

Create a linear model for the demand as it depends on the price. Then, create a corresponding model for the revenue as it relates to the price, where the revenue is the price charged times the number of units sold.

29. Some populations will die off (negative growth rate) unless the size of the population is above some minimum value. Populations that exhibit this behavior are said to have the Allee effect. Let K be the population carrying capacity and M be the minimum threshold for population growth, and assume that $0 < M < K$.

Create a polynomial model for the population's growth rate such that the growth rate is positive for $M < P < K$ and negative for $0 < P < M$ and $P > K$. The growth rate should be zero at $P = 0$, $P = M$, and $P = K$.

30. Optical tweezers (also called laser tweezers) use a tightly focused laser beam to create a force on small transparent objects. The force depends on the location of the object from the center of the beam. When the object is further from the beam than a particular distance, δ, the force pushes the object away. When the distance is less than δ, the force pulls the object toward the center.

Create a polynomial model for the force F as it depends on the position x of the object. Assume that $x = 0$ is the center of the tweezers. The force will have roots at $x = \pm\delta$ and $x = 0$. A positive force pushes the

object to the right and a negative force pushes it to the left. Include a constant factor k and use dimensionless factors for the roots at $x = \pm\delta$.