1.7 Exponents, Roots, and Logarithms

Overview. We frequently think of simplifying expressions as looking for opportunities to cancel terms. We similarly might think of moving terms to the opposite side of an equation, but somehow the opposite term moves to the other side. Although thinking about inverses as opposites can be useful, it is perhaps difficult to generalize properly. A more productive framework with which to think about working with expressions and equations is in terms of operations. We will later generalize the idea of an operation to the mathematical concept of a function. In that sense, inverse operations will correspond to inverse functions.

One of the most common areas relating to inverses where novices encounter trouble is in terms of powers, where the inverse operations are roots and logarithms. For example, a common conceptual error is thinking that logarithms and bases cancel in the same way that factors in fractions cancel. A root is the inverse operation of raising a quantity to a given power. A logarithm is the inverse operation of an exponential operation.

In this section, we review the properties of exponents and focus on distinguishing between the power and exponential operations. We discuss the concept of inverse operations and introduce roots and logarithms as inverses to these operations. We learn to apply inverses to simplify expressions and solve equations involving powers and exponentials.

1.7.1 Properties of Exponents

Do you remember why we have powers as a mathematical notation? When the power is a positive integer, it is to represent repeated multiplication. For example, 3^4 means that we multiply four threes together,

$$3^4 = 3 \cdot 3 \cdot 3 \cdot 3.$$

How did we go from this simple notational convenience to be able to interpret negative powers or fractional powers or even irrational powers? We make sense of these more complex ideas by thinking about what properties the notation should satisfy.

We know that when we multiply and divide by the same number, the net effect is equivalent to multiplying by 1. We say that the terms cancel. We should think of these *actions* as **inverse operations**. That is, the action of multiplying a number by 3 and dividing a number by 3 are inverse operations. If you do them in succession (one after the other), the net effect is equivalent to having applied no operation at all,

$$x \cdot 3 \div 3 = x.$$

Extending this idea allows us to simplify repeated factors represented by powers. How would we simplify $\dfrac{3^5}{3^2}$? If we realize that 3^5 in the numerator means that we multiply by five threes and that the 3^2 in the denominator means that we divide by two threes, then we recognize that there are two pairs of inverse operations:

$$\begin{aligned} \frac{3^5}{3^2} &= \frac{3 \cdot 3 \cdot 3 \cdot 3 \cdot 3}{3 \cdot 3} \\ &= 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \div 3 \div 3 \\ &= 3 \cdot 3 \cdot 3 \cdot (3 \div 3) \cdot (3 \div 3) \end{aligned}$$

$$= 3 \cdot 3 \cdot 3$$
$$= 3^3$$

The net effect of dividing by 3^2 is that we removed two of the threes in the product 3^5. This is why we say that division causes powers to subtract, $\frac{3^5}{3^2} = 3^{5-2} = 3^3$.

All of the basic properties of powers are motivated by the idea that an integer power corresponds to repeated multiplication. For each property, can you think of how it would be a consequence of this idea?

Properties of Powers.

- Zero Power: $b^0 = 1$ for $b \neq 0$
- Inverse Power: $b^{-x} = \frac{1}{b^x}$
- Product with Common Base: $b^x\, b^y = b^{x+y}$
- Quotient with Common Base: $\frac{b^x}{b^y} = b^{x-y}$
- Power of a power: $(b^x)^y = b^{xy}$
- Product with Common Exponent: $b^x\, c^x = (bc)^x$
- Quotient with Common Exponent: $\frac{b^x}{c^x} = \left(\frac{b}{c}\right)^x$

We illustrate several additional properties in the context of integer powers. For integer exponents, a power means repeated multiplication (similar to how multiplication by an integer means repeated addition). So $b^3 = b \cdot b \cdot b$. The product properties are just about counting.

Example 1.7.1

$$b^3 \cdot b^2 = (bbb) \cdot (bb) = b^5 = b^{3+2}$$
$$(b^2)^3 = (bb)(bb)(bb) = b^{2\cdot 3}$$
$$(ab)^3 = (ab)(ab)(ab) = (aaa)(bbb) = a^3 b^3$$

□

The zero power property is necessary for the power of a sum rule to remain consistent. We know that $b^{x+0} = b^x$. But the properties of powers also mean $b^{x+0} = b^x \cdot b^0$. For these to both be true requires $b^x = b^x \cdot b^0$ or that $b^0 = 1$.

The properties of powers relating to products and quotients behave similar to the distributive property of multiplication over addition. This is because multiplication originates as repeated addition, just as powers originate as repeated multiplication. However, addition and powers have no convenient properties. Many mistakes occur when students forget this and imagine that powers distribute over addition like multiplication. (It doesn't!)

Example 1.7.2 To illustrate that $(a+b)^2 \neq a^2 + b^2$, consider the numbers $a = 2$ and $b = 3$. The first expression gives

$$(a+b)^2 = (2+3)^2 = 5^2 = 25$$

while the second expression gives

$$a^2 + b^2 = 2^2 + 3^2 = 4 + 9 = 13.$$

The proper way to expand the first expression is to think of the power as repeated multiplication and apply the distributive property. This is often called the `FOIL` method:

$$(a+b)^2 = (a+b)(a+b) = a^2 + 2ab + b^2.$$

□

1.7.2 Exponent Operations and Their Inverses

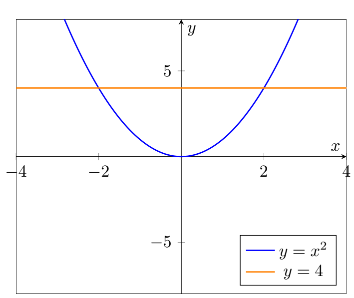
Expressions have a dual interpretation. On the one hand, an expression is a mathematical object that represents some numerical value. On the other hand, an expression also represents a sequence of operations that act on other values. For example, the expression $3x^2 + 5$ is a formula that for each value of x represents a particular value. It also describes a sequence of operations: "Take a value, x. Square it. Multiply by three. Add 5."

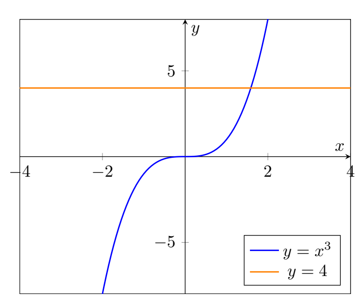
Thinking of powers as operations will require some caution. In the expression, 2^3, the numbers 2 and 3 play different roles. The number 2 is called the **base** and the number 3 is called the exponent. As an action, the expression 2^3, raising 2 to the power 3, can be interpreted in two ways, depending on which number we think of as being acted on. We could say that take the number 2 and apply the power 3. This is likely the more familiar interpretation. Alternatively, we could say that we take the number 3 and apply the base 2. The second interpretation corresponds to the exponential operation.

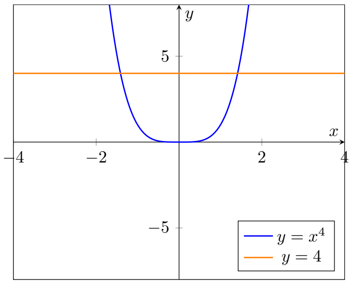
Introducing variables might help make this distinction clearer. This corresponds to thinking of the expression as the value of a function. We will let x represent the number being acted on. The operation of a power 2^3 corresponds to $f(x) = x^3$ with $x = 2$. The same expression interpreted as an exponential operation would be the formula $g(x) = 2^x$ with $x = 3$. An **elementary power function** (applying a power) raises a value to a *fixed power*, $f(x) = x^p$, for a constant p. An **elementary exponential function** (applying a base) uses a value as the exponent of a *fixed base*, $g(x) = b^x$, for a constant b.

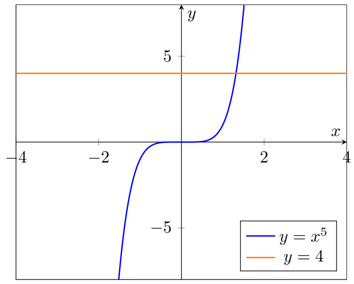
Power and exponential functions have corresponding inverse operations. A root provides the inverse operation to an integer power. A logarithm provides the inverse operation to an exponential. We begin by focusing on powers and roots.

If we wish to solve the equation $x^n = 4$ where n is a positive integer, we might graph $y = x^n$ and $y = 4$ and look for where they intersect.









For even powers n, the equation $x^n = 4$ has two solutions. The graph is symmetric across the y-axis because the product of an even number of negative values is positive, $(-1)^2 = 1$. Notice that if we were trying to solve $x^n = -4$, there would be no solutions when n is even but there would be a single solution when n is odd. The solution is called the nth root.

Definition 1.7.3 For an integer $n > 1$, the nth root $y = \sqrt[n]{x}$ is the value such that $y^n = x$. If n is even, we require $x \geq 0$ and $y \geq 0$. If n is odd, there is no restriction. $\Diamond$

A root can be written as a fractional power. The properties of exponents imply that $(x^p)^n = x^{pn}$. If $p = \frac{1}{n}$, then $(x^{\frac{1}{n}})^n = x$. That is, $x^{\frac{1}{n}} = \sqrt[n]{x}$. This equivalence means that for any rational number $p = \frac{k}{n}$, the power x^p can be computed using integer powers (repeated multiplication) and extracting roots:

$$x^{\frac{k}{n}} = (\sqrt[n]{x})^k.$$

Note 1.7.4 When the base is positive, then the choice of representation in the exponent does not matter. Negative values in the base create complications. For example, we know that a fraction can have multiple representations, like $\frac{1}{3} = \frac{2}{6} = \frac{3}{9}$. Because $(-2)^3 = -8$, we know that $\sqrt[3]{-8} = -2$. This is equivalent to saying $(-8)^{1/3} = -2$. However, $(-8)^{2/6}$ is undefined because the 6th root of -8 is not a real number. On the other hand, it will be true that $(-8)^{3/9} = -2$ as well as for any other equivalent fraction with an odd denominator.

We can simplify expressions that have roots and powers applied consecutively. For example, $\sqrt[3]{a^3}$ takes a value a, and then applies the cubing operation followed by the cube-root operation. Because these are inverses, we recover the original value,

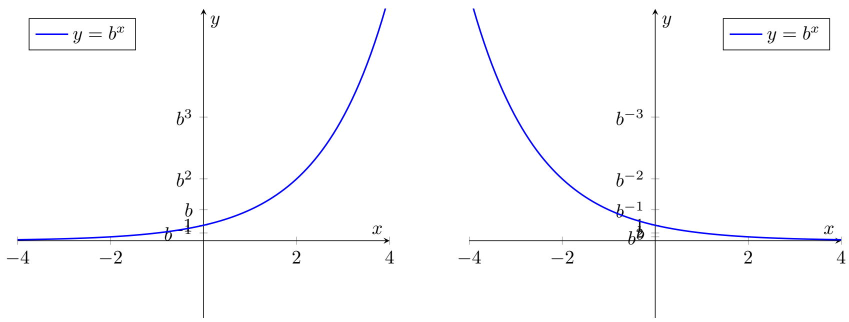
$$\sqrt[3]{a^3} = a.$$

We must be careful, again, when the inverse operations involve even powers. For example, the expression $\sqrt{a^2}$ does not actually simplify to a in all cases. If $a = -2$, then squaring this gives $a^2 = 4$ and then the square root gives $\sqrt{a^2} = \sqrt{4} = 2$. In general, when n is an even power, then $\sqrt[n]{a^n} = |a|$. Applying the power first always makes a^n a positive value, and then the nth root is also defined to return a positive value.

On the other hand, the expression $(\sqrt{x})^2 = x$ rather than $|x|$. In this situation, the first operation is the square root which requires that $x \geq 0$. Squaring the square root of x always recovers the value of x.

How would we define a power with an irrational value? We might approximate the irrational power. That is, we find a rational number that is close to the irrational number and use it instead. This should raise questions. Does our choice of approximation matter? How close do we need to be? Calculus helps us here by introducing the idea of limits. Limits will be central to the ideas of approximation that occur throughout calculus.

Once we know that we can raise any positive base to an arbitrary number as the exponent, we can think of an exponential as a valid operation for a given positive base. Typical graphs of $y = b^x$ for $b > 1$ and for $0 < b < 1$ are shown below. The special case when $b = 1$ corresponds to a horizontal line and is not shown.



The graphs suggest that for any $y > 0$, the equation $b^x = y$ will have a unique solution x for each value y. The solution is called the logarithm of y for the base b.

Definition 1.7.5 For any base b with $b > 0$ and $b \neq 1$, the logarithm with base b of a value $x > 0$ is written $\log_b(x)$. The value $y = \log_b(x)$ is defined for $x > 0$ as that value y such that $b^y = x$. ♢

Notice that both roots and logarithms are defined through the equation that they solve. We can interpret them as operations that will cancel their inverse operations. That is, the consecutive operations of an exponential and a logarithm with the same base cancel one another. In a similar way, a power and its corresponding root cancel one another, although we will have to be careful with even powers because of the even symmetry.

Example 1.7.6 Solve $\sqrt[3]{x} = 2$.

Solution. The equation $\sqrt[3]{x} = 2$ has an isolated cube root on the left. The inverse operation of the cube root is cubing. Starting with a value x, finding its cube root, and then cubing the result just gets back to x. We use this inverse operation in a balanced way to solve the equation.

$$\begin{aligned} \sqrt[3]{x} &= 2 \\ (\sqrt[3]{x})^3 &= 2^3 \\ x &= 2^3 \end{aligned}$$

The solution is $x = 8$. □

Example 1.7.7 Solve $x^4 = 4$.

Solution. The equation $x^4 = 4$ has an isolated integer power on the left. The inverse operation is a fourth root. Because the power is even, there are two solutions.

$$\begin{aligned} x^4 &= 4 \\ \sqrt[4]{x^4} &= \sqrt[4]{4} \\ x &= \pm\sqrt[4]{4} \end{aligned}$$

Because $4 = 2^2$, we could rewrite this as

$$x = \pm(2^2)^{\frac{1}{4}} = \pm 2^{\frac{2}{4}} = \pm 2^{\frac{1}{2}} = \pm\sqrt{2}.$$

□

Example 1.7.8 Solve $\log_3(x) = 2$.

Solution. The equation $\log_3(x) = 2$ has an isolated logarithm. The inverse operation is an exponential with the same base $b = 3$. We use the function representation of this operation, $\exp_3(x) = 3^x$. An equivalent equation is formed by applying this exponential to both sides of the equation.

$$\begin{aligned} \log_3(x) &= 2 \\ \exp_3(\log_3(x)) &= \exp_3(2) = 3^2 \\ x &= 9 \end{aligned}$$

This is saying that the equation $\log_3(9) = 2$ is equivalent to $3^2 = 9$. Notice that we could have just written down the inverse equation immediately, since that is how the logarithm is defined. □

Example 1.7.9 Solve $\log_4(x) = 3$.

Solution. The equation $\log_4(x) = 3$ is defined by the inverse equation $4^3 = x$. So $x = 64$. □

Example 1.7.10 Solve $4^x = 8$.

Solution. The equation $4^x = 8$ has an isolated exponential. The inverse operation is a logarithm with the same base $b = 4$.

$$\begin{aligned} 4^x &= 8 \\ \log_4(\exp_4(x)) &= \log_4(8) \\ x &= \log_4(8) \end{aligned}$$

To go further on this problem, we need more properties.

For this particular problem, we can proceed if we recognize that both 4 and 8 are powers of 2. Because $4 = 2^2$ and $8 = 2^3$, we can rewrite our equation as

$$4^x = 8 \quad \Leftrightarrow \quad 2^{2x} = 2^3.$$

This means that $2x = 3$ or $x = \frac{3}{2}$. We can verify this, since $4^{3/2} = (\sqrt{4})^3 = 2^3 = 8$. That is, $\log_4(8) = \frac{3}{2}$. □

The previous example illustrated a way that we can simplify a logarithm when the base and input are both powers of the same value. In that example, we had $\log_4(8)$ and the base 4 and input 8 were powers of 2. We used the equivalence of the equations

$$x = \log_4(8) \quad \Leftrightarrow \quad 4^x = 8$$

and then rewrote that equation in terms of the common base b to find x.

Example 1.7.11 Simplify $\log_9(\frac{1}{27})$.

Solution. We start by assigning this value to the variable x so that we have an equation,

$$x = \log_9(\frac{1}{27}).$$

The equivalent equation using an exponential instead of a logarithm is

$$9^x = \frac{1}{27}.$$

We recognize that $9 = 3^2$ and $27 = 3^3$ so that the equation can be rewritten

$$3^{2x} = 3^{-3}.$$

This means that $2x = -3$ or $x = -frac32$. That is, $\log_9(\frac{1}{27}) = -\frac{3}{2}$ □

We will learn more general techniques for simplifying logarithms later. Most scientific calculators only have logarithms for base $b = 10$ and for base $b = e$. The logarithm for $b = 10$ is called the common logarithm and appears on a calculator with out a base `log`. The logarithm for $b = e$ is called the natural logarithm and appears on a calculator as `ln`. We will later prove that every logarithm can be found using one of these by the change of base formula

$$\log_b(x) = \frac{\log(x)}{\log(b)} = \frac{\ln(x)}{\ln(b)}.$$

Example 1.7.12 Solve $3^x = 5$.

Solution. The unknown x has an exponential with base $b = 3$ operation acting on it. To isolate the variable, we need to apply the inverse operation to both sides.

$$\begin{aligned} 3^x &= 5 \\ \log_3(3^x) &= \log_3(5) \\ x &= \log_3(5) \end{aligned}$$

We have solved the equation, but we don't have a good sense of what that number might be. We know that $3^1 = 3$ and $3^2 = 9$, so $x = \log_3(5)$ must be somewhere between 1 and 2. The change of base formula allows us to approximate the value on a calculator,

$$x = \log_3(5) = \frac{\ln(5)}{\ln(3)} \approx 1.46497.$$

□

1.7.3 Applications

We encountered equations that require roots and logarithms to solve when we considered exponential and power function models for data. Recall that a **general power function** has the form

$$f(x) = A\,x^p,$$

where A and p are the model parameters. A **general exponential function** has the form

$$f(x) = A\,b^x,$$

where A and b are the model parameters, with $b > 0$ and $b \neq 1$.

Example 1.7.13 A colony of bacteria is observed to cover an area of $2.5\,\text{mm}^2$. Six hours later, the colony has expanded to cover a space of $100\,\text{mm}^2$. Assuming that the bacteria is growing according to an exponential growth model, develop a model for the area of the colony as a function of the time since the first observation. If the population continues to follow this model, at what time will the bacteria colony fill a dish with $6000\,\text{mm}^2$?

Solution. An exponential model relates the area of the colony C (mm^2) as a function of time t (hours) according to the formula

$$C = A\,b^t,$$

where A and b are model parameters to be determined by the data. The first observation, $t = 0$ and $C = 2.5$, corresponds to a parameter equation

$$2.5 = A\,b^0.$$

The second observation, $t = 6$ and $C = 100$, corresponds to a parameter equation

$$100 = A\,b^6.$$

Because $b^0 = 1$, the first equation gives $A = 2.5$. Substituting this into the second equation gives an equivalent equation

$$100 = 2.5\,b^6.$$

We solve the equation for b by isolating the variable. The expression currently has a product with 2.5, so we apply the inverse operation of dividing by 2.5 on both sides,

$$\frac{100}{2.5} = b^6.$$

The expression now has a power operation, and the inverse is a root,

$$b = \left(\frac{100}{2.5}\right)^{1/6} = \sqrt[6]{40} \approx 1.84931.$$

Consequently, our model for the colony area is given by

$$C = 2.5 \cdot 1.84931^t.$$

To answer the final question, we see that t is unknown but $C = 6000$. Substituting this into the model equation, we obtain an equation only involving t,

$$6000 = 2.5 \cdot 1.84931^t.$$

To isolate the variable, we first need to divide by 2.5,

$$\frac{6000}{2.5} = 1.84931^t.$$

Now, the variable has an exponential operation acting on it. The inverse operation is a logarithm with base $b = 1.84931$, so that

$$t = \log_{1.84931}(2400) = \frac{\ln(2400)}{\ln(1.84931)} \approx 12.6595.$$

The model predicts that the bacteria colony will fill the dish after abou 12.66 hours, which is approximately 12 hours and 40 minutes. □

Example 1.7.14 In the 1930s, a Swiss biologist named Max Kleiber observed that the metabolic rate of mammals approximately follows a power law relation with the mass of the animal. Let Q represent the average metabolic rate (in kJ per day) and let M represent the average mass (in kg). A mouse has an average mass $M = 0.021$ kg and an average metabolic rate of $Q = 20.9$ kJ/day. A horse has an average mass $M = 400$ kg and an average metabolic rate of $Q = 32000$ kJ/day. Find a power law model that matches this data. Use the model to predict the metabolic rate for a cat which has an average mass $M = 3$ kg.

Solution. We start with the model equation, $Q = A\,M^p$, with model parameters A and p. Using the data allows us to create a system of equations for our parameters.

$$\begin{aligned} (M,Q) = (0.021, 20.9) \quad &\Rightarrow \quad 20.9 = A\,0.021^p \\ (M,Q) = (400, 32000) \quad &\Rightarrow \quad 32000 = A\,400^p \end{aligned}$$

To solve a system of equations, we solve one equation for one variable. For this problem, we use the first equation to solve for A:

$$20.9 = A\,0.021^p \qquad \Leftrightarrow \qquad A = \frac{20.9}{0.021^p}.$$

We now substitute this expression in place of A into the other equation.

$$\begin{aligned} 32000 &= \frac{20.9}{0.021^p}\,400^p \\ 32000 &= 20.9\frac{400^p}{0.021^p} = 20.9\left(\frac{400}{0.021}\right)^p \end{aligned}$$

To solve for the remaining unknown, we need to apply inverse operations. The current expression involves multiplication by 20.9, so the inverse operation is division by 20.9.

$$\frac{32000}{20.9} = \left(\frac{400}{0.021}\right)^p$$

Now the expression is an exponential of p with base $b = \frac{400}{0.021}$. The inverse operation is the logarithm,

$$\log_b\left(\frac{32000}{20.9}\right) = p.$$

We use the change of base formula to find the decimal approximation,

$$p = \frac{\ln(32000/20.9)}{\ln(400/0.021)} \approx 0.744187.$$

Knowing p, we go back to find the value for A,

$$A = \frac{20.9}{0.021^p} \approx 370.45.$$

Our approximate power law model is therefore $Q = 370.45 \cdot M^{0.744187}$. Using the average mass of a cat $M = 3$, we predict

$$Q = 370.45 \cdot 3^{0.744187} \approx 839.067.$$

The observed metabolic rate for cats is actually $Q = 546$ kJ/day, which is lower than predicted. This should not disappoint us too much, as Kleiber's law was really based on a regression model for many animals. Some values will be above the model's prediction and some will be below. □

1.7.4 Summary

- Properties of exponents are motivated by the idea that integer powers correspond to repeated multiplication.
- Exponential functions (exponential growth or exponential decay) have the form $f(x) = A \cdot b^x$, with the independent variable in the exponent.
- Power functions have the form $f(x) = A \cdot x^p$, with the independent variable as the base of the power.
- Roots, such as the square root or cube root, are inverse operations for the power operation:

$$x^n = y \qquad \Leftrightarrow \qquad x = \sqrt[n]{y}.$$

When n is an even integer, we require $x \geq 0$ and $y \geq 0$.

- Roots can be written as reciprocal powers:

$$\sqrt[n]{x} = x^{1/n}.$$

- Logarithms are inverse operations for the exponential operation, defined for every base $b > 0$ and $b \neq 1$:

$$b^x = y \qquad \Leftrightarrow \qquad x = \log_b(y).$$

- The change of base rule allows us to find decimal approximations for any base using the common or natural logarithm:

$$\log_b(x) = \frac{\log(x)}{\log(b)} = \frac{\ln(x)}{\ln(b)}.$$

1.7.5 Exercises

Identify a property of exponents to rewrite an equivalent expression. Note that each property can be applied in either direction.

1. 3^{x+2}

2. $(2x)^3$

3. $2^x \cdot 3^x$

4. $\dfrac{x^3}{4^3}$

5. $\dfrac{3^u}{3^4}$

6. 2^{3x}

7. A student made a mistake writing $3 \cdot 2^x = 6^x$. What did the student do? Why was it incorrect?

8. A student made a mistake writing $2^x \cdot 3^y = 6^{x+y}$. What did the student do? Why was it incorrect?

Find an exact value for each root or logarithm. Do not use a calculator.

9. $\sqrt{9a^2}$

10. $\sqrt[3]{-8a^6}$

11. $\log_2(8)$

12. $\log_2(\frac{1}{8})$

13. $\log_3(\frac{1}{81})$

14. $\log_3(1)$

15. $\log_4(2)$

16. $\log_4(32)$

17. $\log_{1/2}(4)$

18. $\log_{1/4}(2)$

19. $\log_8(\frac{1}{2})$

20. $\log_8(16)$

21. $\log_{25}(\frac{1}{125})$

Solve the equations.

22. $x^7 = 4$

23. $3x^3 = 8$

24. $\sqrt[4]{x} = 3$

25. $3\sqrt[3]{2x} = 4$

26. $5^x = 10$

27. $3^{2x} = 4$

28. $\log_4(x) = 2$

29. $\log_3(2x) = 9$

30. $4\log_5(x) = 15$

31. $3x^2 - x^6 = 0$

32. $4 \cdot x^5 = 3$

33. $4 \cdot 5^x = 3$, writing solutions in terms of the natural logarithm `ln`.

34. $2 \cdot 3^{-x} = 3 \cdot 2^x$, writing solutions in terms of the natural logarithm `ln`. Hint: Find an equivalent equation with a single exponential after using properties of exponents.

Applications.

35. Find a power law for y as a function of x that includes data $(x, y) = (1, 5)$ and $(x, y) = (4, 10)$.

36. Find a power law for y as a function of x that includes data $(x, y) = (2, 20)$ and $(x, y) = (4, 5)$.

37. Find a power law for y as a function of x that includes data $(x, y) = (2, 4)$ and $(x, y) = (20, 8)$.

38. Find an exponential law for y as a function of x that includes data $(x, y) = (0, 5)$ and $(x, y) = (4, 15)$.

39. Find an exponential law for y as a function of x that includes data $(x, y) = (0, 5)$ and $(x, y) = (10, 2)$.

40. Find an exponential law for y as a function of x that includes data $(x, y) = (1, 5)$ and $(x, y) = (6, 10)$.

41. The average body mass and life span of mammals have been observed to follow an approximate power law. A mouse has an average body mass of 0.021 kg and a life span of 1.5 years. A horse has an average body mass of 400 kg and a life span of 40 years. Find a power function model for the life span as a function of body mass. Predict the life span of a typical hare, which has a body mass of 3.4 kg.

42. The fraction of carbon in organic matter that is radioactive (carbon-14) decays exponentially from the time of death. At the time of death, the fraction of radiocarbon would be 1.25 parts per trillion. A sample that is 1000 years old has a fraction of radiocarbon measured at 1.1075 parts per trillion. Model the fraction in parts per trillion as an exponential function of time since death. Estimate the age of a sample that has radiocarbon measured at 0.8 parts per trillion.

43. P-32 is a radioactive isotope of phosphorus used in labeling biological molecules. P-32 has a half-life of 14.29 days. Suppose an experiment begins with 10 µg of P-32. Find a parametrized model for the mass (in µg) as a function of time (in days) measured from the start of the experiment, $t \mapsto M$, in order to determine how much P-32 remains after 10 days and after 100 days.

Hint. Create two constraints using $t = 0$ and $t = 14.29$.

44. The isotope of plutonium Pu-239 has a half-life of 24,110 years, which is the time after which half of the mass has decayed. For an initial mass of 1 kg, how much plutonium remains after 100 years?

45. An exponentially growing population that doubles in size every 5 years currently has 1000 individuals.

(a) What will the population be in 4 years?

(b) How long does it take for the population to triple?