# 1.8 Logarithms and Their Properties

We previously learned about the algebraic properties of exponents. Because logarithms are the inverses of exponential operations, they inherit related algebraic properties. The properties of logarithms allow us to rewrite expressions involving products and exponents in new ways.

In this section, we will introduce the properties of logarithms and how they relate to the properties of exponents. These properties are closely related to the logarithmic scale. We will learn new methods for solving equations involving exponentials using the properties of the logarithm. In addition, we will learn how to rewrite exponential functions in terms of other bases and justify the change of base formula for logarithms.

## 1.8.1 The Logarithmic Scale

Historically, the logarithm was invented so that calculations involving multiplication and division could be solved using the much simpler operations of addition and subtraction. Addition and multiplication share many of the same properties—commutativity, associativity, and the existence of an identity and inverses. These shared properties suggest that there might be a way to think about multiplication in terms of addition.

The properties of exponents establish the relationship required to make this connection. When two numbers can be written as powers of the same base, say $u = b^x$ and $v = b^y$, then the product $uv$ can be written as a power of that base as $uv = b^{x+y}$. Division $u \div v$ can similarly be written as a power, $u \div v = b^{x-y}$. In this way, multiplication and division of numbers directly corresponds to addition and subtraction of their associated powers for a given base.

The logarithm relates a number and its associated power for a given base. For every number $u > 0$ and base $b > 0$ with $b \neq 1$,

$$u = b^x \qquad \Leftrightarrow \qquad x = \log_b(u).$$

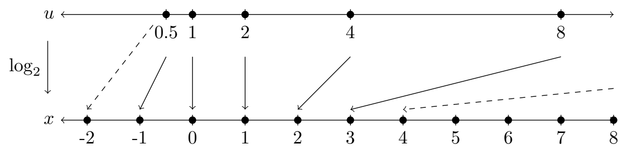
We will try to develop an intuition by creating the map between a number and its logarithm. In the process, we will develop something called a logarithmic number line or logarithmic scale.

A map is a way of thinking about the relationship between two variables. We use one variable as the independent variable, or input, for the map. This is the value we start with. The relation tells us that this value for the independent variable is associated with a particular value of the dependent variable. The value of the dependent variable is the output of the map.

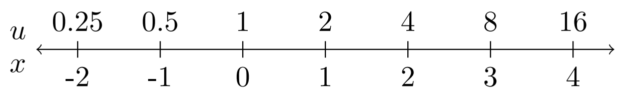
To make our example precise, we will create the map for a base $b = 2$. However, the process could be done for any base $b > 0$ with $b \neq 1$. With a base $b = 2$, we find $\log_2(u)$ by solving the equation $2^x = u$. The logarithm is the map $u \mapsto x$. The equation is easy to solve when $u$ is a known power of 2.

$$\begin{aligned}
1 = 2^0 \quad &\Leftrightarrow (u, x) = (1, 0) \quad \Leftrightarrow \quad \log_2(1) = 0 \\
2 = 2^1 \quad &\Leftrightarrow (u, x) = (2, 1) \quad \Leftrightarrow \quad \log_2(2) = 1 \\
4 = 2^2 \quad &\Leftrightarrow (u, x) = (4, 2) \quad \Leftrightarrow \quad \log_2(4) = 2 \\
8 = 2^3 \quad &\Leftrightarrow (u, x) = (8, 3) \quad \Leftrightarrow \quad \log_2(8) = 3 \\
\tfrac{1}{2} = 2^{-1} \quad &\Leftrightarrow (u, x) = (\tfrac{1}{2}, -1) \quad \Leftrightarrow \quad \log_2(\tfrac{1}{2}) = -1
\end{aligned}$$

These simple logarithms are illustrated in the following map.




Now, instead of drawing two number lines, let us use a single number line labeling the points with the input above the line and the output below the line.




This relation could be extended to *every* point on the number line. The value above the number line is just the value $2^x$ where $x$ is the value below the number line. Some special values should be explicitly identified. The point with $x = \frac{1}{2}$ corresponds to $u = 2^{1/2} = \sqrt{2} \approx 1.4142$. Similarly, the point with $x = \frac{3}{2}$ corresponds to $u = 2^{3/2} = \sqrt{8} \approx 2.8284$.

The resulting locations of numbers above the number line is called a **logarithmic scale**. A more detailed logarithmic scale is provided below. Because consecutive integers are closer and closer together in a logarithmic scale, the figure only shows the tick mark location for some of the values.

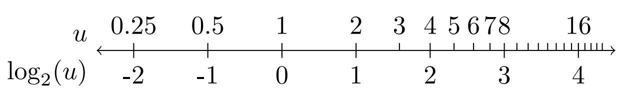



**Figure 1.8.1** Logarithmic scale showing the logarithm base two, $b = 2$.

In our example, we developed the logarithmic scale using a base $b = 2$. We could have done it with any base. The logarithmic scale with base $b = 10$, corresponding to the common logarithm, is shown below.

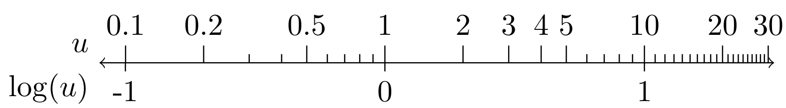



**Figure 1.8.2** Logarithmic scale showing the common logarithm, $b = 10$.

Comparing the logarithmic scales for bases $b = 2$ and $b = 10$, you should notice that the logarithmic scale itself appears to be the same. The only thing that is different is the scale of the tick marks below the number line corresponding to the value of the logarithm.

## 1.8.2 Properties of the Logarithm

When considering the logarithmic scales, we always had $\log_b(1) = 0$. This is because $b^0 = 1$ for every valid base. This creates a mapping from the multiplicative identity $u = 1$ to the additive identity $x = 0$. That is, the value $u = 1$ becomes the origin of the logarithmic scale.

In a similar way, because $b^1 = b$, the logarithm will always have $\log_b(b) = 1$. Be careful that you do not think of this as a cancellation. Instead, think of the logarithmic scale. The value $u = 1$ sets the origin. The logarithm measures the position of each number as if using a ruler starting at $u = 1$. The value $u = b$ sets the length scale for the ruler.

The core properties of logarithms are summarized below.

**Theorem 1.8.3 Properties of Logarithms.** *For every base $b > 0$ with $b \neq 1$ and values $u > 0$ and $v > 0$, logarithms satisfy the following properties.*

- *Identities Map:* $\log_b(1) = 0$.
- *The Base Sets the Scale:* $\log_b(b) = 1$.
- *Inverse Property:* $b^{\log_b(u)} = u$ *and* $\log_b(b^x) = x$.
- *Product Rule:* $\log_b(u \cdot v) = \log_b(u) + \log_b(v)$.
- *Quotient Rule:* $\log_b\left(\dfrac{u}{v}\right) = \log_b(u) - \log_b(v)$.
- *Power Rule:* $\log_b(u^p) = p \cdot \log_b(u)$.

*Proof.* The first two properties were proved in the paragraphs preceding the theorem. For the remaining three properties, because $u > 0$, we know that $b^x = u$ has a solution $x = \log_b(u)$. Similarly, for $v > 0$, we know that there is a value $y = \log_b(v)$ so that $b^y = v$. The inverse property is simply stating what it means to be a logarithm.

The power rule considers the logarithm of the value $u^p$. By rewriting $u = b^x$, we are looking for the logarithm of $u^p = (b^x)^p$. By the Power of a Power property, we have $u^p = b^{(xp)}$, which means that $\log(u^p) = xp = p \cdot \log_b(u)$.

The product rule considers the logarithm of the value $u \cdot v$. By rewriting $u = b^x$ and $v = b^y$, we are looking for the logarithm of $uv = b^x \cdot b^y$. By the Product with a Common Base property, we have $uv = b^{x+y}$, which means that $\log(uv) = x + y = \log_b(u) + \log_b(v)$. The proof of the quotient rule for logarithms is proved in a similar way. ■

The properties of logarithms allow us to compute the logarithm of a product (and quotient) in terms of the logarithms of the individuals factors.

**Example 1.8.4** A reference table shows $\log(2) \approx 0.30103$, $\log(3) \approx 0.47712$ and $\log(5) \approx 0.69897$. Use properties of logarithms to determine $\log(1.2)$.

**Solution**. Start by writing 1.2 as a product of the factors 2, 3, and 5.

$$1.2 = \frac{12}{10} = \frac{4(3)}{2(5)} = \frac{2(3)}{5}$$

The properties of logarithms allow us to rewrite $\log(1.2)$ as

$$\begin{aligned}
\log(1.2) &= \log\left(\frac{2(3)}{5}\right) \\
&= \log(2(3)) - \log(5) && \text{(Quotient)} \\
&= \log(2) + \log(3) - \log(5) && \text{(Product)} \\
&\approx 0.30103 + 0.47712 - 0.69897 = 0.07918.
\end{aligned}$$

□

Historically, the logarithm was invented so that multiplication and division calculations could be solved using the much simpler operations of addition and subtraction. Engineers and scientists would often reference logarithm tables to find the logarithms of the factors and add the values by hand. Then they would look in the table for the number that had the resulting logarithm. The slide rule was a mechanical implementation, where the lengths corresponding to logarithm values were added physically to get a new length. For an interactive demonstration, check out http://educ.jmu.edu/~waltondb/webapp/log_scale_explore.html.

Although modern calculators and computers have eliminated this particular need for logarithms in calculations, we still use logarithms in order to expand symbolic formulas expressed as products and powers in terms of sums of the logarithms of the factors. To expand a logarithm, we look at the structure of the expression to which the logarithm is applied. There are likely many parts, but we look at the final operation that is used to form that expression.

1. If the input of the logarithm is a *product* of expressions, then we expand the logarithm by rewriting it as a sum of logarithms each with one of the factors as input.

2. If the input of the logarithm is a *quotient* of expressions, then we expand the logarithm by rewriting it as a difference of logarithms, adding the logarithm of the numerator and subtracting the logarithm of the denominator.

3. If the input of the logarithm is a *power* applied to an expression, then we expand the logarithm by rewriting it as the value of the power times the logarithm of the base.

4. If the input of the logarithm is a *anything else*, then the rules of logarithms do not apply. In particular, we can not expand the logarithm of a sum.

By changing any quotients into multiplication by negative powers, we can reduce the process of expanding logarithms to only two rules: the product and power rules.

**Example 1.8.5** Expand $\log\left(\dfrac{4x^3\sqrt{2x+5}}{(x^2+3)^5}\right)$ as far as possible.

**Solution**. Each factor of the expression inside the logarithm will get its own term using the product and quotient rules for logarithms. If we think of every division in terms of negative powers, then we only need to deal with products. In particular, the inner expression can be rewritten

$$\frac{4x^3\sqrt{2x+5}}{(x^2+3)^5} = 4x^3(2x+5)^{\frac{1}{2}}(x^2+3)^{-5}.$$

The factors identified are $4$, $x^3$, $(2x+5)^{\frac{1}{2}}$, and $(x^2+3)^{-5}$. Using the logarithm's product rule followed by the logarithm's power rule, we find

$$\begin{aligned}\log\left(\frac{4x^3\sqrt{2x+5}}{(x^2+3)^5}\right) &= \log(4) + \log(x^3) + \log\left((2x+5)^{\frac{1}{2}}\right) + \log\left((x^2+3)^{-5}\right)\\ &= \log(4) + 3\log(x) + \tfrac{1}{2}\log(2x+5) - 5\log(x^2+3).\end{aligned}$$

Notice that we could have used the quotient rule of logarithms instead of negative powers to get the term $-5\log(x^2+3)$. □

The rules for expanding logarithms can also be used in reverse to collect terms into a single logarithm.

**Example 1.8.6** Collect the terms of

$$2 + 4\log x + 2\log y - \frac{1}{2}\log(x+y)$$

to write the expression in terms of a single logarithm.

**Solution**. We first recognize that every logarithm times an expression can be rewritten as a logarithm with the expression in the exponent.

$$2 + 4\log x + 2\log y - \frac{1}{2}\log(x+y) = 2 + \log(x^4) + \log(y^2) + \log\left((x+y)^{-1/2}\right)$$

Now that the expression is a sum of logarithms. What about the isolated number 2? Using the common logarithm with base $b = 10$, we know $\log(10^2) = 2$.

$$\begin{aligned} 2 + 4\log x + 2\log y - \frac{1}{2}\log(x+y) &= 2 + \log(x^4) + \log(y^2) + \log\left((x+y)^{-1/2}\right) \\ &= \log(10^2\, x^4\, y^2\, (x+y)^{-1/2}) \\ &= \log\left(\frac{100x^4y^2}{\sqrt{x+y}}\right) \end{aligned}$$

□

### 1.8.3 Using Logarithms to Solve Equations

The properties of logarithms help solve us equations, particularly where the variable is in an exponent. Logarithm and exponential operations are invertible, so we can use them as balanced operations on an equation to obtain new equivalent equations. The properties of the logarithm can then be used to our advantage.

**Example 1.8.7** Solve the equation $3 \cdot 2^{3x} = 5$ using the logarithm base 10.

**Solution**. It is usually best to isolate the exponential term first.

$$\begin{aligned} 3 \cdot 2^{3x} &= 5 \\ 2^{3x} &= \frac{5}{3} \end{aligned}$$

We next apply the logarithm base 10 to both sides of this equation, which then allows us to apply the logarithm power rule on the left. Then we can isolate $x$.

$$\begin{aligned} \log_{10}(2^{3x}) &= \log_{10}(\tfrac{5}{3}) \\ 3x \cdot \log_{10}(2) &= \log_{10}(\tfrac{5}{3}) \\ x &= \frac{\log_{10}(\frac{5}{3})}{3\log_{10}(2)} \end{aligned}$$

Alternatively, we could have applied the logarithm at the very first. This would require using the logarithm product rule on the left.

$$\begin{aligned} \log_{10}(3 \cdot 2^{3x}) &= \log_{10}(5) \\ \log_{10}(3) + \log_{10}(2^{3x}) &= \log_{10}(5) \\ \log_{10}(2^{3x}) &= \log_{10}(5) - \log_{10}(3) \\ 3x\log_{10}(2) &= \log_{10}(5) - \log_{10}(3) \\ x &= \frac{\log_{10}(5) - \log_{10}(3)}{3\log_{10}(2)} \end{aligned}$$

□

The properties of logarithms allow us to compute logarithms with uncommon bases using logarithms that we know using the change of base formula.

**Theorem 1.8.8** *For any two exponential bases $b$ and $c$,*

$$\log_b(u) = \frac{\log_c(u)}{\log_c(b)}.$$

*Proof.* Consider $x = \log_b(u)$, which solves $b^x = u$. If we solve this equation using $\log_c$ to both sides, we find the change of base formula.

$$\begin{aligned} b^x &= u \\ \log_c(b^x) &= \log_c(u) \\ x\log_c(b) &= \log_c(u) \\ x = \log_b(u) &= \frac{\log_c(u)}{\log_c(b)}. \end{aligned}$$

■

This theorem has a nice geometric interpretation in relation to the logarithmic scale. The value $\log_b(u)$ measures the distance in the logarithmic scale from the origin at 1 to the location of $u$ in terms of the scale set by the location of $b$. The change of base formula is based on knowing the positions of $u$ and $b$ on the logarithmic scale in terms of the base $c$. We take the position of $u$ and divide it by the length to get to $b$. In effect, this is a change of units calculation, similar to finding a distance measured in inches when we know the distance measured in centimeters.

Closely related to the change of base formula is the fact that we can rewrite any power with a positive base using a composition with an elementary exponential. We will soon discover that the number $e$ is the natural exponential base. Thus, every power can be rewritten in terms of the natural exponential function. The logarithm with this base is the **natural logarithm** ln.

**Example 1.8.9** Rewrite $f(x) = 4 \cdot 3^{2x}$ using the natural exponential function.

**Solution**. One approach is to use the inverse property, $e^{\ln(u)} = u$, on the factor with the power. Then use logarithm properties to simplify (expand) the power.

$$\begin{aligned} f(x) &= 4 \cdot 3^{2x} \\ &= 4 \cdot e^{\ln(3^{2x})} && \text{(Inverse Property)} \\ &= 4 \cdot e^{2x\ln(3)} && \text{(Power Rule)} \\ &= 4 \cdot e^{2\ln(3)\,x} && \text{(Commute)} \end{aligned}$$

Another approach is to just rewrite the base using the inverse property, $3 = e^{\ln(3)}$, and then finish by using properties of powers.

$$\begin{aligned} f(x) &= 4 \cdot 3^{2x} \\ &= 4 \cdot (e^{\ln(3)})^{2x} && \text{(Inverse Property)} \\ &= 4 \cdot e^{2x\ln(3)} && \text{(Power of Product)} \\ &= 4 \cdot e^{2\ln(3)\,x} && \text{(Commute)} \end{aligned}$$

□

We use base $e$ for exponentials so much that we summarize the statement as a theorem.

**Theorem 1.8.10** *Every exponential function $f(x) = Ab^x$ can be written using the natural exponential $f(x) = Ae^{kx}$ where $k = \ln(b)$.*

**Example 1.8.11** A population $P$ is an exponential function of time $t$, $P = Ae^{kt}$. Suppose that $P = 500$ when $t = 0$ and the population triples every 5 years. Find the formula for $P$.

**Solution**. This is an exponential model $P = Ae^{kt}$ with unknown parameters $A$ and $k$. We use the data $(t, P) = (0, 500)$ and $(t, P) = (5, 1500)$ (the population triples in 5 years) to create equations based on our model that constrain the parameters.

$$\begin{cases} 500 = Ae^{0k} \\ 1500 = Ae^{5k} \end{cases}$$

The first equation gives $A = 500$. Substituting that into the second equation, we can solve for $k$.

$$\begin{aligned} Ae^{5k} &= 1500 \\ 500e^{5k} &= 1500 \\ e^{5k} &= 3 \\ 5k &= \ln(3) \\ k &= \tfrac{1}{5}\ln(3) \end{aligned}$$

Using these parameters, we have our model

$$P = 500e^{\frac{1}{5}\ln(3)t}.$$

□

### 1.8.4 Summary

- The logarithmic scale is based on the inverse map of an exponential function. The multiplicative identity $u = 1$ represents the origin of the scale. The base $b$ establishes the length scale used to calculate logarithms.
- The properties of logarithms are consequences of properties of exponents.
  - $\log_b(1) = 0$ (Identities Map) and $\log_b(b) = 1$ (Base Sets Scale).
  - $b^{\log_b(u)} = u$ and $\log_b(b^x) = x$ (Inverse Property).
  - $\log_b(uv) = \log_b(u) + \log_b(v)$ (Product Rule).
  - $\log_b(u \div v) = \log_b(u) - \log_b(v)$ (Quotient Rule).
  - $\log_b(u^p) = p \cdot \log_b(u)$ (Power Rule).
- To expand a logarithm is to identify if the input of a logarithm is a product, quotient, or power, and then to rewrite that logarithm as a sum of logarithms, a difference of logarithms, or a product with a logarithm. This is repeated until no logarithm has an input that is a product, quotient, or power.

### 1.8.5 Exercises

**1.** For an unknown base $b$, we have $\log_b(2) = 0.3$. Use the rules of logarithms to find each of the following.

(a) $\log_b(8)$

(b) $\log_b(\frac{1}{\sqrt{2}})$

(c) $\log_b(4b^2)$

Can you identify the value $b$?

**2.** For an unknown base $b$, we have $\log_b(2) \approx 0.3562$ and $\log_b(3) \approx 0.5646$. Use the rules of logarithms to find each of the following.

(a) $\log_b(6)$

(b) $\log_b(72)$

(c) $\log_b(\frac{4}{9})$

**3.** Expand $\log\left(3x^5(2x+1)^4\right)$ as far as possible.

**4.** Expand $\log\left(\dfrac{(x^2+4)^3}{x^4(3x+1)}\right)$ as far as possible.

**5.** Expand $\log\left(\sqrt{5(x^2+1)}\right)$ as far as possible.

**6.** Rewrite the expanded formula $2 + 3\log(x) - \log(2x+1)$ as the logarithm (base 10) of a single expression.

**7.** Rewrite the expanded formula $\frac{1}{2}\ln(x) - \ln(x+1) - \ln(x-1)$ as the logarithm (base e) of a single expression.

**8.** Use the natural logarithm to solve the equation $4 \cdot 5^x = 3$.

**9.** Use the natural logarithm to solve the equation $2 \cdot 3^{-x} = 3 \cdot 2^x$.

**10.** Write the function $f(x) = 4^x$ using an exponential with base 10.

**11.** Write the function $f(x) = 4^x$ using an exponential with base $e$.

**12.** Write the function $f(x) = 5 \cdot 0.25^x$ using an exponential with base $e$.

**13.** Write the function $f(x) = x^4$ using an exponential with base $e$ for $x > 0$.

**14.** Write the function $f(x) = x^{2x}$ using an exponential with base $e$ for $x > 0$.

**15.** Find an exponential model $y = Ae^{kx}$ satisfying the states $(x, y) = (0, 3)$ and $(x, y) = (5, 9)$.

**16.** Find an exponential model $y = Ae^{kx}$ satisfying the states $(x, y) = (1, 3)$ and $(x, y) = (4, 6)$.

**17.** In a living organism, 1 gram of carbon would result in about 840 carbon-14 atoms disintegrating per hour. After death, the rate of radiocarbon disintegrations decays exponentially. Carbon-14 has a half-life of 5730 years, meaning the rate has dropped to half its original value after this time. Determine the radioactive disintegration rate for 1 gram of carbon using the natural exponential base $e$. What is the radioactive disintegration rate of the sample after 1000 years?

**18.** A money market account starting at \$2000 grew by 10% in one year. Determine the value of the money market account assuming the rate of growth remains constant by using an exponential growth model. What will be the value in 4 years? How long does it take for the value to double?