12.5 Continuous Functions

12.5.1 Continuity

Recall our definition of continuity for a function at a single point.

Definition 12.5.1 Continuity at a Point. A function f is **continuous** at a if

$$\lim_{x \to a} f(x) = f(a).$$

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The single equation captures the full definition because for the equation to be true, the limit must exist and the value of the function must exist. Also, recall that the function is **right-continuous** if the limit comes from the right ($x \to a^+$) and **left-continuous** if the limit comes from the left ($x \to a^-$).

These ideas allow us to define what we mean by saying that a function is continuous on an interval.

Definition 12.5.2 Continuity on an Interval. A function f is **continuous on an interval** (a, b) if f is continuous at every point $x \in (a, b)$. We can include an endpoint if the limit statement is true coming from within the interval. That is, we include a if

$$\lim_{x \to a^+} f(x) = f(a)$$

and we include b if

$$\lim_{x \to b^-} f(x) = f(b).$$

◊

12.5.2 Definite Integrals and Average Value

When we studied the definite integral, we learned that continuity implies integrability. However, a discontinuous function might still be integrable. For example, the definite integral of a piecewise continuous function with a finite number of jump discontinuities can be computed using the splitting property. The total definite integral would be equal to the sum of the definite integrals on each of the subintervals.

Continuity does guarantee something stronger than integrability. It guarantees that the function attains its average value over an interval. To make this precise, we first need to define the average value.

Definition 12.5.3 Average Value of a Function. The **average value** of a function f on an interval $[a, b]$, denoted $\langle f \rangle_{[a,b]}$, is defined as

$$\langle f \rangle_{[a,b]} = \frac{1}{b-a} \int_a^b f(x)\, dx,$$

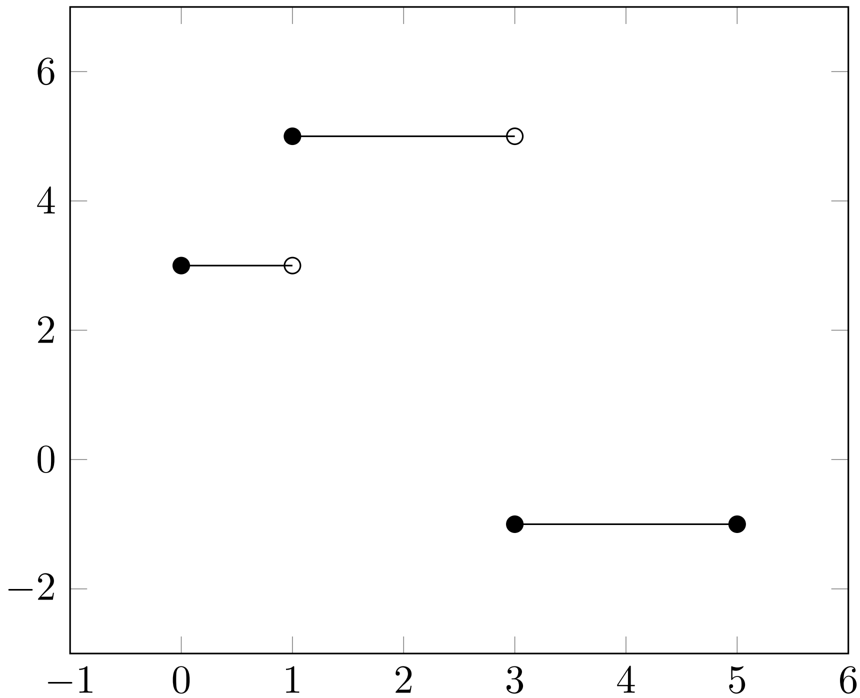
so long as f is integrable on $[a, b]$. ◊

The average value is defined as the value of a constant function that has the same definite integral over the interval:

$$\int_a^b \langle f \rangle_{[a,b]}\, dx = \langle f \rangle_{[a,b]} \cdot (b-a) = \int_a^b f(x)\, dx..$$

Example 12.5.4 The figure below illustrates a simple function $f(x)$ defined on the interval $[0,5]$,

$$f(x) = \begin{cases} 3, & 0 \le x < 1, \\ 5, & 1 \le x < 3, \\ -1, & 3 \le x \le 5. \end{cases}$$

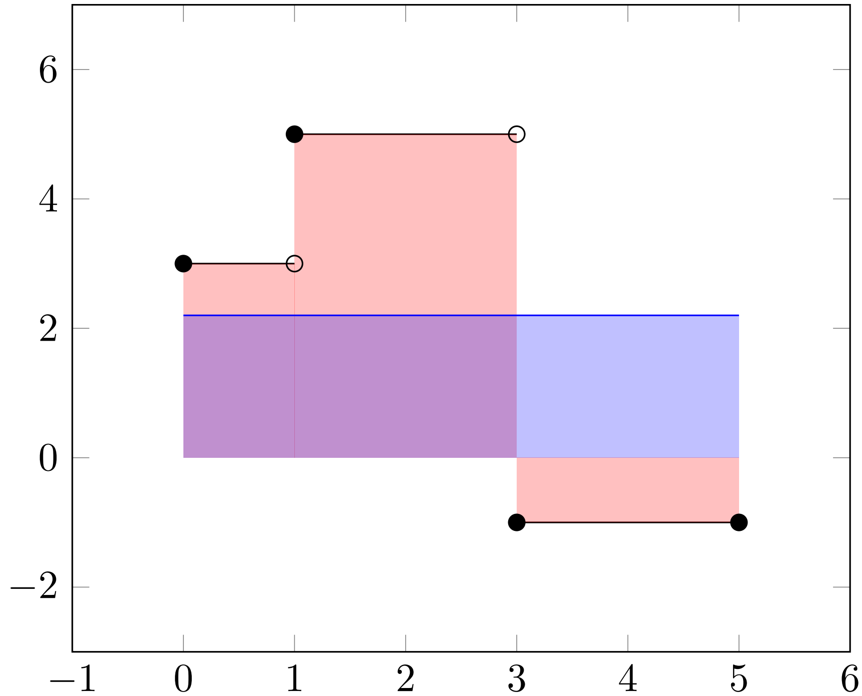


The definite integral equals the sum of the signed areas,

$$\int_0^5 f(x)\,dx = 3 \cdot 1 + 5 \cdot 2 + -1 \cdot 2 = 11.$$

The average value is equal to this definite integral divided by the width of the interval,

$$\langle f \rangle_{[0,5]} = \frac{1}{5} \int_0^5 f(x)\,dx = \frac{11}{5}.$$



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Theorem 12.5.5 Mean Value Theorem for Integrals. *Given a function f that is continuous on $[a,b]$, there must exist a value $c \in (a,b)$ such that*

$$f(c) = \langle f \rangle_{[a,b]} = \frac{1}{b-a}\int_a^b f(x)\,dx,$$

or equivalently, $\displaystyle\int_a^b f(x)\,dx = f(c)\cdot(b-a)$.

Proof. Because f is continuous on $[a,b]$, the Extreme Value Theorem guarantees that f attains a minimum value $f(x_{\text{min}})$ and a maximum value $f(x_{\text{max}})$ so that $f(x_{\text{min}}) \le f(x) \le f(x_{\text{max}})$ for all $x \in [a,b]$.

The average value $\langle f \rangle_{[a,b]}$ must be between the minimum and maximum values. The (((Unresolved xref, reference "thm-integral-inequality"; check spelling or use "provisional" attribute)))Integral Bounds theorem guarantees

$$f(x_{\text{min}})(b-a) \le \int_a^b f(x)\,dx \le f(x_{\text{max}})(b-a)$$

which then implies

$$f(x_{\text{min}}) \le \langle f \rangle_{[a,b]} \le f(x_{\text{max}}).$$

By the Intermediate Value Theorem with the interval with end points x_{min} and x_{max} (we don't know which is on the left/right), there must be some value c between these points, and so $c \in (a,b)$, for which

$$f(c) = \langle f \rangle_{[a,b]}.$$

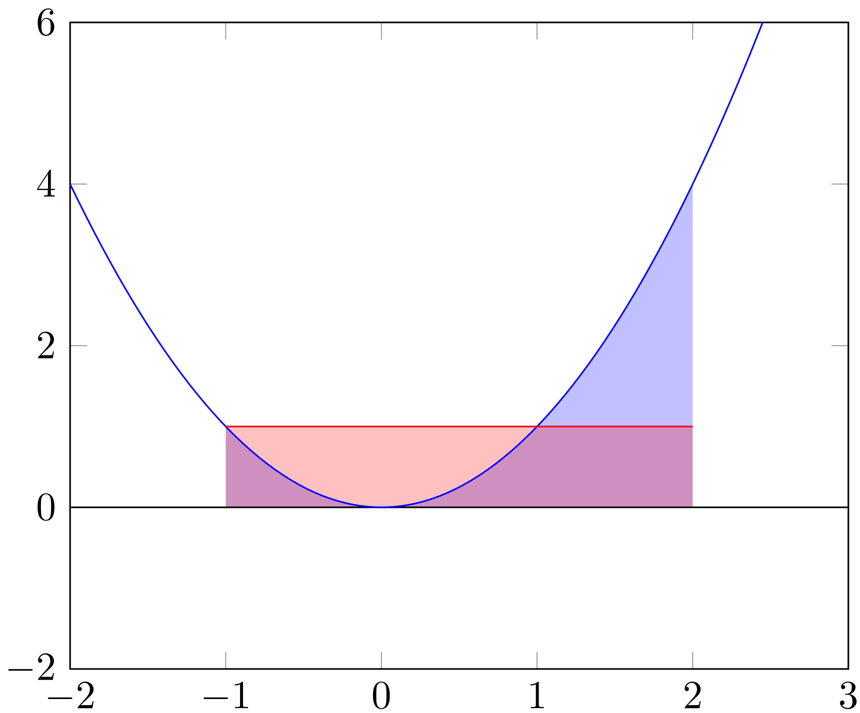
■

In the previous example, f was not continuous and we can see that the graph $y = f(x)$ did not intersect the constant value $\langle f \rangle_{[0,5]}$. The Mean Value Theorem for Integrals guarantees that when the function is continuous, the constant function using the average value must intersect the graph $y = f(x)$.

Example 12.5.6 The function $f(x) = x^2$ is continuous everywhere. The average value on the interval $[-1, 2]$ can be found using the (((Unresolved xref, reference "thm-elementary-definite-integrals"; check spelling or use "provisional" attribute)))elementary accumulation formula for a quadratic rate and the splitting property.

$$\begin{aligned}\langle f\rangle_{[-1,2]} &= \frac{1}{2 - -1}\int_{-1}^{2} x^2\,dx \\ &= \frac{1}{3}\Big(\int_0^2 x^2\,dx - \int_0^{-1} x^2\,dx\Big) \\ &= \frac{1}{3}\Big(\frac{1}{3}(2^3) - \frac{1}{3}(-1)^3\Big) = \frac{1}{3}\Big(\frac{8}{3} + \frac{1}{3}\Big) = 1\end{aligned}$$

A figure showing the graphs $y = f(x) = x^2$ and $y = \langle f\rangle_{[-1,2]} = 1$ is shown below. The Mean Value Theorem predicted the existence of a point $c \in (-1, 2)$ where $f(c) = \langle f\rangle_{[-1,2]} = 1$, which we can see occurs at $c = 1$.



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The Mean Value Theorem for Integrals also provides the justification for the Monotonicity Test for Accumulation Functions.

Theorem 12.5.7 Monotonicity Test for Accumulation Functions. *Suppose that $A(x)$ is an accumulation function with corresponding rate function $f(x)$, and suppose that $f(x)$ is continuous on $[a, b]$.*

- *If $f(x) > 0$ for all $x \in (a, b)$, then $A(x)$ is increasing on $[a, b]$.*
- *If $f(x) < 0$ for all $x \in (a, b)$, then $A(x)$ is decreasing on $[a, b]$.*
- *If $f(x) = 0$ for all $x \in (a, b)$, then $A(x)$ is constant on $[a, b]$.*

Proof. Consider any two points $c_1, c_2 \in [a, b]$ with $c_1 < c_2$. Because $A(x)$ is an accumulation function, by the splitting property of definite integrals,

$$A(c_2) - A(c_1) = \int_{c_1}^{c_2} f(x)\,dx.$$

On the other hand, because f is continuous, the Mean Value Theorem guarantees the existence of a point $c \in (c_1, c_2)$ such that

$$A(c_2) - A(c_1) = \int_{c_1}^{c_2} f(x)\,dx = f(c) \cdot (c_2 - c_1).$$

Now assume that $f(x) > 0$ for all $x \in (a, b)$. Then $f(c) > 0$ and $c_2 - c_1 > 0$, guaranteeing that $A(c_2) - A(c_1) > 0$. That is, $A(c_2) > A(c_1)$. This is what is needed to show that A is increasing.

Next assume that $f(x) < 0$ for all $x \in (a, b)$. Then $f(c) < 0$ while $c_2 - c_1 > 0$, guaranteeing that $A(c_2) - A(c_1) < 0$. That is, $A(c_2) < A(c_1)$, which shows that A is decreasing.

Finally assume that $f(x) = 0$ for all $x \in (a, b)$. Then $f(c) = 0$, implying that $A(c_2) - A(c_1) = 0$. That is, $A(c_2) = A(c_1)$, which shows that A is constant. ■