

12.6 Applications Involving Densities

12.6.1 Overview

Having developed the theory of definite integrals and functions defined as the accumulation on increments, we turn our attention to applications of these ideas. One of the most common mathematical applications is the calculation of area of regions bounded by curves. Physically, this is closely related to the calculation of total mass and center of mass. The same calculations are used in statistics to calculate probabilities and averages.

The general setup for many applications involving definite integrals is to think of the total quantity as a sum of the parts. If a region is cut into separate pieces, for example, then the area of the total region should be the sum of the areas of each region measured separately. Quantities that have this property are called **extensive**. A definite integral can be used to compute extensive quantities if we can consider the total as a sum of small increments over a partition of an independent variable.

Theorem 12.6.1 Using Definite Integrals to Compute Extensive Quantities. *Suppose an extensive quantity Q can be subdivided into increments corresponding to a uniform partition of an independent variable x over an interval $[a, b]$. If there is a function $f(x)$ so that on each subinterval $[x_{k-1}, x_k]$ there is some point x_k^* so that*

$$f(x_k^*)\Delta x = \Delta Q_k,$$

then

$$Q = \int_a^b f(x) dx.$$

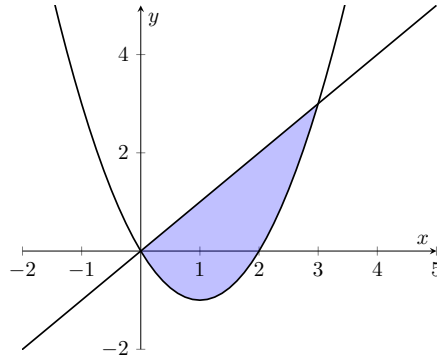
In this context, the rate of accumulation function f used to compute the quantity Q as an integral is often called the **density** of Q with respect to x .

This section will explore the application of definite integrals to compute various extensive quantities. This will require imagining a partition, identifying the independent variable and its corresponding interval for integration, determining the appropriate function used as the density, and setting up the definite integral. Because our emphasis will be on identifying the appropriate definite integral, we will use technology to compute the the resulting value.

12.6.2 Area of Regions in the Plane

When we developed the definite integral of a function f on an interval $[a, b]$, we noted that the integral represented the total signed area over the interval. It was signed because the accumulation of negative values when the function was below the axis was subtracting area. Because area is an extensive quantity, it is an ideal example of a quantity that can be computed using integration.

Example 12.6.2 Find the area of the region bounded between $y = x^2 - 2x$ and $y = x$.



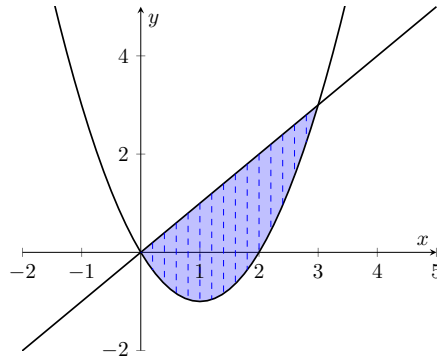
Solution. Start by identifying a convenient variable to partition. The region is determined by where the graphs $y = x^2 - 2x$ and $y = x$ intersect. Using substitution and solving, the intersection occurs at the solution to $x^2 - 2x = x$.

$$x^2 - 2x = x$$

$$x^2 - 3x = 0$$

$$x(x - 3) = 0$$

The region is completely contained between $x = 0$ and $x = 3$. We choose our independent variable to be x over the interval $[0, 3]$. Now, imagine a partition of the interval and consider the increments of area over each subinterval.



As the increments Δx are smaller and smaller, the increment of area ΔA will be closely approximated by the width Δx times the vertical distance between $y = x$ and $y = x^2 - 2x$,

$$\Delta A \approx (x - (x^2 - 2x)) \Delta x.$$

The total area is the sum of the increments, so we can use a definite integral. The height $h(x) = x - (x^2 - 2x) = 3x - x^2$ between the curves acts as the density of area,

$$A = \int_0^3 (x - (x^2 - 2x)) dx = \int_0^3 3x - x^2 dx.$$

The density is a simple polynomial so that we can compute the value using the elementary accumulation formulas.

$$\begin{aligned} \int_0^3 3x - x^2 dx &= 3 \int_0^3 x dx - \int_0^3 x^2 dx \\ &= 3\left(\frac{1}{2}(3)^2\right) - \left(\frac{1}{3}(3)^3\right) \end{aligned}$$

$$\begin{aligned}
 &= \frac{27}{2} - 9 \\
 &= \frac{9}{2}
 \end{aligned}$$

Except for such simple problems, we can use technology to compute or approximate the value of the integrals. The SageMath engine attempts to compute integrals exactly using the `integrate` command, which uses the following syntax.

```
integrate(3*x-x^2, [x,0,3])
```

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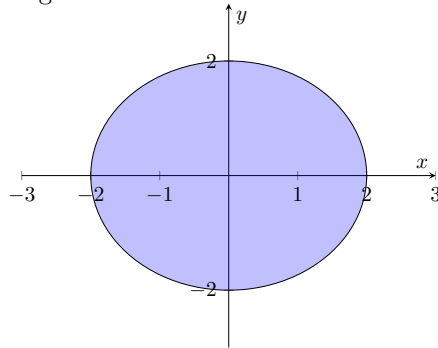
When SageMath is unable to do the exact calculation, we can still do a numerical approximation using the `numerical_integral` command.

```
numerical_integral(3*x-x^2, 0, 3)
```

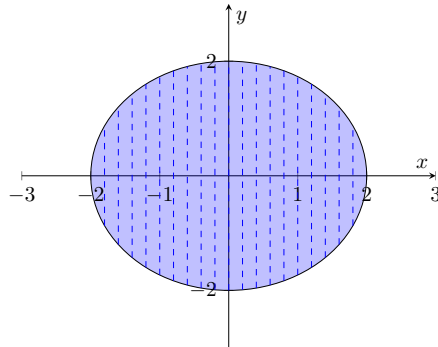
(4.5, 4.9960036108132044e-14)

The result gives an approximate answer along with an estimated error bound. In this case, we find $A = 4.5 \pm 4.996 \times 10^{-14}$. \square

Example 12.6.3 Express the area of a circle with center $(0,0)$ and radius $r = 2$ as a definite integral.



Solution. The circle is clearly between the lines $x = -2$ and $x = 2$ so that we can imagine a partition with variable x on the interval $[-2, 2]$. The increments of area ΔA correspond to thin vertical strips with width Δx (from the partition) and a height computed as the distance from the top of the circle to the bottom of the circle.



The equation of the circle is

$$x^2 + y^2 = 4.$$

To find the height of the increments, we need to know the two y -values for each x -value,

$$y = \pm\sqrt{4 - x^2}.$$

The height is the difference between the values,

$$h(x) = (\sqrt{4 - x^2}) - (-\sqrt{4 - x^2}) = 2\sqrt{4 - x^2}.$$

Consequently, the area of the circle is defined by the integral

$$A = \int_{-2}^2 h(x) dx = \int_{-2}^2 2\sqrt{4 - x^2}.$$

Computational tools can compute this value, which is consistent with the known formula

$$A = \pi r^2 = \pi(2)^2 = 4\pi.$$

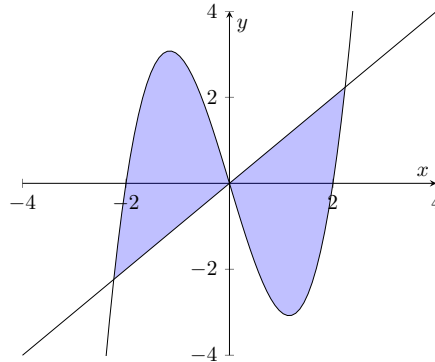
```
integrate(2*sqrt(4-x^2), [x, -2, 2])
```

4*pi

□

When curves cross multiple times, we may need to compute the area of individual regions.

Example 12.6.4 Find the area bounded by the graphs $y = x$ and $y = x^3 - 4x$.



Solution. We start by identifying the points of intersection of the curves by solving the equation $x^3 - 4x = x$.

$$x^3 - 4x = x$$

$$x^3 - 5x = 0$$

$$x(x^2 - 5) = 0$$

One solution is $x = 0$, corresponding to the intersection at $(x, y) = (0, 0)$. Two other solutions come from $x^2 = 5$ at $x = \pm\sqrt{5}$. The total area consists of two regions, the first with $x \in [-\sqrt{5}, 0]$ and the second with $x \in [0, \sqrt{5}]$.

On the first interval $[-\sqrt{5}, 0]$, the height of increments is given by

$$h(x) = (x^3 - 4x) - x = x^3 - 5x$$

because the cubic polynomial is the top curve. So the area over the interval $[-\sqrt{5}, 0]$ is computed by

$$A_1 = \int_{-\sqrt{5}}^0 x^3 - 5x dx.$$

In order to use the elementary accumulation formulas, the integral needs to start at $x = 0$, so we reverse the order of integration and change the sign.

$$\begin{aligned}
 A_1 &= \int_{-\sqrt{5}}^0 x^3 - 5x \, dx \\
 &= - \int_0^{-\sqrt{5}} x^3 - 5x \, dx \\
 &= - \int_0^{-\sqrt{5}} x^3 \, dx + 5 \int_0^{-\sqrt{5}} x \, dx \\
 &= - \left(\frac{1}{4} (-\sqrt{5})^4 \right) + 5 \left(\frac{1}{2} (-\sqrt{5})^2 \right) \\
 &= -\frac{25}{4} + \frac{25}{2} \\
 &= \frac{25}{4}
 \end{aligned}$$

On the second interval $[0, \sqrt{5}]$, the height of increments is given by

$$h(x) = x - (x^3 - 4x) = 5x - x^3$$

because now the cubic polynomial is the bottom curve. The corresponding area is

$$A_2 = \int_0^{\sqrt{5}} 5x - x^3 \, dx$$

which has a value

$$\begin{aligned}
 A_2 &= \int_0^{\sqrt{5}} 5x - x^3 \, dx \\
 &= 5 \int_0^{\sqrt{5}} x \, dx - \int_0^{\sqrt{5}} x^3 \, dx \\
 &= 5 \left(\frac{1}{2} (\sqrt{5})^2 \right) - \left(\frac{1}{4} (\sqrt{5})^4 \right) \\
 &= \frac{25}{2} - \frac{25}{4} \\
 &= \frac{25}{4}
 \end{aligned}$$

As we should expect from symmetry, the two areas are equal $A_1 = A_2$.

The total area of the region is

$$A = A_1 + A_2 = \frac{25}{4} + \frac{25}{4} = \frac{25}{2}.$$

If we were to think of the distance between the two curves in terms of the absolute value, the integral could be computed over a single interval,

$$A = \int_{-\sqrt{5}}^{\sqrt{5}} |x - (x^3 - 4x)| \, dx = \int_{-\sqrt{5}}^{\sqrt{5}} |5x - x^3| \, dx.$$

Using SageMath, the absolute value prevents an exact integral.

```

A1 = integrate(x^3-5*x, [x,-sqrt(5),0])
A2 = integrate(5*x-x^3, [x,0,sqrt(5)])
show(A1)
show(A2)
show(A1+A2)

```

25/4
25/4
25/2

```
integrate(abs(5*x-x^3), [x,-sqrt(5),sqrt(5)])
```

```
integrate(abs(5*x-x^3), [x,-sqrt(5),sqrt(5)])
```

```
numerical_integral(abs(5*x-x^3), -sqrt(5),sqrt(5))
```

```
(12.5, 1.3855583347321954e-13)
```

□

12.6.3 Density and Mass

12.6.4 Summary

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12.6.5 Exercises

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