5.6 The Fundamental Theorem of Calculus, Part One

5.6.1 Overview

When we introduced the definite integral, we also learned about accumulation functions. An accumulation function is a function A defined as a definite integral from a fixed lower limit a to a variable upper limit where the integrand is a given function f,

$$A(x) = A(a) + \int_a^x f(z)\,dz.$$

The function f was called the rate of accumulation for the function A, and we wrote $A'(x) = f(x)$. Then we defined another rate of change, the instantaneous rate of change, with a corresponding function called the derivative. For a function $F(x)$, the derivative was defined by a limit,

$$\frac{dF}{dx}(x) = \lim_{h\to 0} \frac{F(x+h) - F(x)}{h}.$$

This section establishes a relation between these two concepts of the rate of change. The Fundamental Theorem of Calculus proves that the derivative of an accumulation function exactly matches the rate of accumulation at whenever the rate of accumulation is continuous. That is, the instantaneous rate of change of a quantity, which graphically gives the slope of the tangent line on the graph, is exactly the same as the value of the rate of accumulation when the function is expressed as an accumulation using a definite integral. The proof of the fundamental theorem relies on properties of continuous functions as well as properties of limits.

5.6.2 Illustration of an Example

To illustrate the concept that we will prove, let us consider a simple polynomial function

$$f(x) = x^3 - 3x + 5.$$

Using our rules of accumulation, we know that $f(x)$ can be written as an accumulation,

$$f(x) = 5 + \int_0^x 3z^2 - 3\,dz.$$

What happens if we compute the derivative using the definition?

We start with some preparatory algebra based on $f(x) = x^3 - 3x + 5$.

$$\begin{aligned} f(x+h) &= (x+h)^3 - 3(x+h) + 5 \\ &= (x+h)(x+h)(x+h) - 3(x+h) + 5 \\ &= (x^2 + 2xh + h^2)(x+h) - 3x - 3h + 5 \\ &= x^3 + 3x^2h + 3xh^2 + h^3 - 3x - 3h + 5 \end{aligned}$$

$$\begin{aligned} f(x+h) - f(x) &= (x^3 + 3x^2h + 3xh^2 + h^3 - 3x - 3h + 5) - (x^3 - 3x + 5) \\ &= 3x^2h + 3xh^2 + h^3 - 3h \\ &= h(3x^2 + 3xh + h^2 - 3) \end{aligned}$$

The derivative can be computed using the limit:

$$\begin{aligned}\frac{df}{dx}(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\ &= \lim_{h\to 0}\frac{h(3x^2+3xh+h^2-3)}{h}\\ &= \lim_{h\to 0} 3x^2+3xh+h^2-3\\ &= 3x^2+3x(0)+(0)^2-3\\ &= 3x^2-3.\end{aligned}$$

The derivative exactly matches the rate of accumulation.

Our ultimate goal in this section is to show that this will always happen for accumulation functions. To reach this goal, we require some additional concepts.

5.6.3 Average Value of a Function

The derivative computes the limit of the average rate of change. In preparation for the Fundamental Theorem of Calculus, we need a relation between the idea of average value and the definite integral.

Consider how we compute the average value of a list of numbers. We add up all the values and divide by the number of values in the list. When thinking about a function f on an interval $[a,b]$, there are infinitely many different values to consider. We need a different way to think about it. We define the average value of a function using a limit of a standard average value.

Let f be a piecewise continuous function on $[a,b]$ so that the definite integral $\int_a^b f(x)\,dx$ will be defined. Consider a uniform partition of the interval $[a,b]$ with $\Delta x = \frac{b-a}{n}$ and $x_k = a + k\cdot\Delta x$, just as we defined when creating a Riemann sum. We *approximate* the average value of f on the interval $[a,b]$, which is represented by the symbol $\langle f\rangle_{[a,b]}$, by finding the average of the values $f(x_k)$ for $k=1,2,\ldots,n$:

$$\langle f\rangle_{[a,b]} \approx \frac{1}{n}\sum_{k=1}^{n} f(x_k).$$

The approximation is improved with larger and larger values of n, so the actual average value will be the limit of the average as $n\to\infty$:

$$\langle f\rangle_{[a,b]} = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} f(x_k).$$

The average value defined by this limit looks remarkably similar to the limit of a Riemann sum that would define a definite integral. In particular,

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{k=1}^{n} f(x_k)\frac{b-a}{n}.$$

Comparing our two limits reveals that the definite integral is the same as the average value multiplied by $b-a$. This will be our formal definition of the average value of a function.

Definition 5.6.1 For a function $f(x)$ which has an integral on an interval

$[a, b]$, the **average value** of $f(x)$ on $[a, b]$ is defined by

$$\langle f(x) \rangle_{[a,b]} = \frac{1}{b-a} \int_a^b f(x)\, dx.$$

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When we think of the definite integral as the total signed area between the graph $y = f(x)$ and the axis $y = 0$, the average value can be interpreted as the value of a constant function that would have the same signed area. This is a consequence of writing

$$\int_a^b f(x)\, dx = (b-a) \cdot \langle f(x) \rangle_{[a,b]}.$$

The average value multiplied by $b-a$ equals the total signed area of $f(x)$ from a to b, so we can think of $(b-a) \cdot \langle f(x) \rangle_{[a,b]}$ as the signed area of a rectangle with a vertical position given by the average value.

Imagine the region below the graph $y = f(x)$ and between the lines $x = a$ and $x = b$ as if it were frozen water. If the ice melted but was trapped between the vertical lines $x = a$ and $x = b$, the high regions of the ice would melt and fill the valleys until the water level was flat. The resulting flat value is equivalent to the average value of the original function. Any regions of area above the average value line are used to fill an equivalent amount of area missing below the line.

Example 5.6.2 Consider finding the average value of $f(x) = x^2 - 2$ on the interval $[0, 2]$. The average value is computed by dividing the definite integral of $f(x) = x^2 - 2$ as x goes from 0 to 2 by the length of the interval.

$$\begin{aligned}
\langle x^2 \rangle_{[0,2]} &= \frac{1}{2-0} \int_0^2 x^2 - 2\, dx \\
&= \frac{1}{2} \Big[\frac{1}{3} x^3 - 2x \Big]_0^2 \\
&= \frac{1}{2} \Big(\frac{8}{3} - 2(2) \Big) - \frac{1}{2} \Big(\frac{0}{3} - 2(0) \Big) \\
&= \frac{1}{2} \Big(\frac{-4}{3} \Big) = -\frac{2}{3}
\end{aligned}$$

Having found the average value, we now create a graph of $y = f(x) = x^2 - 2$ together with the graph $y = \langle f(x) \rangle_{[0,2]} = -\frac{2}{3}$, as shown in Figure 5.6.3. The graph of the average value is the horizontal line. We can see that the area above the average value is matched by the unshaded area below the average value line and above the function.

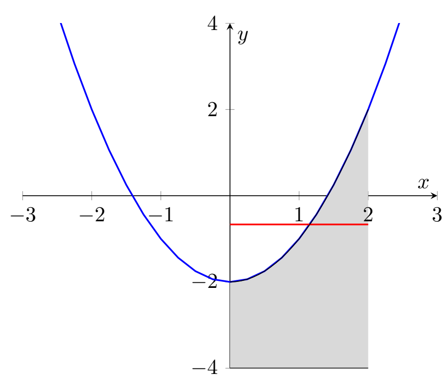

Figure 5.6.3 The graph of $y = x^2 - 2$ along with its average value over $[0, 2]$.

□

We need one more theorem before we discuss the Fundamental Theorem. That theorem is called the Mean Value Theorem for Definite Integrals. The phrase **mean value** is equivalent to **average value**, just as the mean of a set of numbers is equivalent to the average of those numbers. We have previously pointed out that the average or mean value of a function over an interval is equal to the constant value (horizontal function) that would have the same integral.

In Figure 5.6.3, we can see that the function actually crosses the line representing average value. As an equation, this point of intersection corresponds to a solution of the equation

$$f(x) = \langle f(x) \rangle_{[0,2]} \qquad \Leftrightarrow \qquad x^2 - 2 = -\frac{2}{3}.$$

When a function is continuous, such an intersection point will always occur. If a function is not continuous, it is possible for the function to skip over its average value without such an intersection.

Theorem 5.6.4 Mean Value Theorem for Definite Integrals. *If a function f is continuous on $[a, b]$, then there is a value $c \in (a, b)$ so that*

$$f(c) = \langle f \rangle_{[a,b]} = \frac{1}{b-a} \int_a^b f(x)\, dx.$$

Equivalently, the equation can be rewritten

$$f(c) \cdot (b - a) = \int_a^b f(x)\, dx.$$

Proof. Because f is continuous on $[a, b]$ (a closed interval), the Theorem 5.4.14 guarantees that f has an absolute maximum value f_{max} and an absolute minimum value f_{min} inside the interval. The average value must be between the maximum and minimum values. Indeed, for all $x \in [a, b]$ we have

$$f_{\text{min}} \leq f(x) \leq f_{\text{max}}.$$

Definite integrals preserve the ordering so that

$$f_{\text{min}} \cdot (b - a) \leq \int_a^b f(x)\, dx \leq f_{\text{max}} \cdot (b - a),$$

and dividing by the length of the interval $b - a$ we have

$$f_{\text{min}} \leq \langle f(x) \rangle_{[a,b]} \leq f_{\text{max}}.$$

The Theorem 5.4.18 then guarantees that between the x-values where the extremes occur we must have at least one value $x = c$ where $f(x) = \langle f(x) \rangle_{[a,b]}$. ■

The Mean Value Theorem for Definite Integrals will give us a tool with which we can replace a definite integral by a corresponding value of the integrand or rate function at some point within the interval times the length of that interval.

5.6.4 The Fundamental Theorem of Calculus

Now, are you ready to be blown away? Having learned what the average value of a function over an interval represents—the constant height that would give the same signed area over the interval— we can discover that there is a relationship between the ideas of average rate of change and average value.

Let's think about an accumulation function, A with its corresponding rate of accumulation A'. What is the average rate of change of $A(x)$ as x goes from a to b?

$$\frac{\Delta A}{\Delta x}\Big|_{a,b} = \frac{A(b) - A(a)}{b - a} = \frac{1}{b - a}\left(A(b) - A(a)\right).$$

Because A is an accumulation, the change ΔA can be rewritten using an integral and

$$\frac{\Delta A}{\Delta x}\Big|_{a,b} = \frac{1}{b-a}\int_a^b A'(x)\,dx.$$

But that is just the average value of the rate of accumulation function A'. Two completely different ideas of average value end up measuring the very same thing.

Theorem 5.6.5 *The average rate of change (the difference quotient) of an accumulation function $A(x)$ is exactly equal to the average value (the integral divided by interval length) of the corresponding rate of accumulation $A'(x)$:*

$$\frac{\Delta A}{\Delta x}\Big|_{a,b} = \langle A' \rangle_{[a,b]}.$$

Pay attention, however, that we are thinking about two different functions. The average value of the rate of accumulation is based on the integral of the rate $A'(x)$. The average rate of change uses the rate of change based on the difference quotient using $A(x)$. The equivalence of these two averages provides exactly what is necessary to compute the derivative of an accumulation function.

Theorem 5.6.6 The Fundamental Theorem of Calculus, Part One (FTC1). *Given any function $f(x)$ that is continuous on an interval I and a value $a \in I$. The accumulation function*

$$A(x) = \int_a^x f(z)\,dz$$

is differentiable and $\frac{dA}{dx}(x) = f(x)$ for all $x \in I$.

Proof. Given the accumulation function $A(x)$ and its associated integrand $f(x)$, we consider the average rate of change of A between x and $x+h$. By Theorem 5.6.5, we can rewrite this in terms of the value of the rate function f,

$$\frac{\Delta A}{\Delta x}\Big|_{x,x+h} = \frac{A(x+h) - A(x)}{h} = f(c_h),$$

for some value c_h between x and $x+h$. The symbol c_h includes the h to emphasize that this value depends on h.

Now consider a sequence of values $h \to 0$. Because c_h is between x and $x+h$ for all h, the corresonding sequence c_h also converges, $c_h \to x$. As f is a continuous function,

$$\frac{dA}{dx}(x) = \lim_{h\to 0}\frac{\Delta A}{\Delta x}\Big|_{x,x+h} = \lim_{h\to 0} f(c_h) = f(x).$$

■

What have we shown? Earlier, when discussing accumulation functions,

$$A(x) = A(a) + \int_a^x f(z)\,dz,$$

we learned to *identify* the rate function based on its appearance within the definite integral and we wrote $A'(x) = f(x)$ as a way of describing this association. In that association, the prime (apostrophe) of A' was telling us to *identify the appropriate rate of accumulation*.

Now we have learned about another concept, the derivative, which is defined as the limiting value of the average rate of change of a function f from the input value of interest x and a second point $x+h$ as $h \to 0$. This is a fundamentally different concept from accumulation defined as the limit of a Riemann sum. Nevertheless, when we compute the derivative of an accumulation function, we recover exactly that function's corresponding rate of accumulation, so long as the rate of accumulation is a continuous function.

The rate of accumulation and the derivative are really different perspectives of the same function. This surprisingly deep relationship between definite integrals and derivatives will continue to develop.

5.6.5 Summary

- For simple polynomials which we previously learned to express as accumulation functions, the rate of accumulation seems miraculously to agree with the derivative of the polynomial.

- The Fundamental Theorem of Calculus (FTC1) shows us that this isn't circumstance but will always happen when the rate of accumulation is a continuous function. That is, the derivative of an accumulation function will equal the corresponding rate function. When we write $f'(x)$, that means *both* the rate of accumulation when $f(x)$ is an accumulation function *and* the derivative of a function $f(x)$ because those are ultimately the same thing.

- We can compute the **average value of a function** on an interval $[a,b]$ using a definite integral,
$$\langle f(x) \rangle_{[a,b]} = \frac{1}{b-a} \int_a^b f(x)\, dx.$$
The integral replaces the idea of adding a list of values and dividing by the length of the interval replaces the idea of dividing by the number of values being added.

- The average rate of change of an accumulation function $f(x)$ and the average value of the rate of accumulation $f'(x)$ for that function are equal to each other.

- The Mean Value Theorem for Definite Integrals guarantees that for a continuous function, the equation $f(c) = \langle f(x) \rangle_{[a,b]}$ has a solution for some value $c \in (a,b)$. It allows us to substitute
$$\int_a^b f(x)\, dx = f(c) \cdot (b-a)$$
for some c between a and b, but does not tell us how to find c.

5.6.6 Exercises

Find the average value of the given function over the given interval. Sketch a graph of the function and the average value over the interval. Then solve for c so that $f(c) = \langle f(x) \rangle_{[a,b]}$, if it exists.

1. $f(x) = 4 + 2x$ with $[a, b] = [0, 2]$.

2. $f(x) = 4x - x^2$ with $[a, b] = [0, 2]$.

3. $f(x) = 4x - x^2$ with $[a, b] = [0, 4]$.

4. $f(x) = \begin{cases} 1, & 0 \le x < 2 \\ 3, & 2 \le x \le 3 \end{cases}$ with $[a, b] = [0, 3]$. You will need to split the integral into two intervals. (Hint: Think about the graph geometrically.)

5. $f(x) = \begin{cases} 2x, & 0 \le x < 2 \\ 6 - x, & 2 \le x \le 4 \end{cases}$ with $[a, b] = [0, 4]$. You will need to split the integral into two intervals. (Hint: Think about the graph geometrically.)

6. $f(x) = \begin{cases} x, & 0 \le x < 2 \\ 5 - x, & 2 \le x \le 3 \end{cases}$ with $[a, b] = [0, 3]$. You will need to split the integral into two intervals. (Hint: Think about the graph geometrically.)

Applying the Fundamental Theorem of Calculus.

7. Find $\frac{dF}{dx}(x)$ where $F(x) = 10 + \int_1^x z^2 - 3z\, dz$. Then give the equation of the tangent line at $x = 1$.

8. Find $\frac{dG}{dx}(x)$ where $G(x) = \int_0^x \frac{1}{z^2 + 4}\, dz$. Then give the equation of the tangent line at $x = 0$.

9. Find $\frac{dH}{dx}(x)$ where $H(x) = 3 + \int_2^x z e^{-z^2}\, dz$. Then give the equation of the tangent line at $x = 2$.

Applications of Average Value

10. The density (kilograms per meter) of a rod that is two meters long depends on position along the rod according to the equation

$$\rho(x) = 2 - 0.25x, \quad 0 \le x \le 2.$$

Find the average density of the rod.

11. A car accelerates from 0 to 64 miles per hour over eight seconds so that the velocity of the car is a function of time given by

$$v(t) = 16t - t^2, \qquad 0 \le t \le 8.$$

What is the average velocity of the car during those eight seconds? How far does the car travel? (Hint: Use Theorem 5.6.5 and pay attention to units.)

12. During a rainstorm, the rate R (inches per hour) at which rain fell varied according to the following relation,

$$R(t) = \begin{cases} 2t, & 0 \le t < 0.25, \\ 0.5, & 0.25 \le t < 0.5, \end{cases}$$

where t is measured in hours. What is the average rate at which rain fell during the storm? What was the total amount of rain that fell during the storm? (Hint: Use Theorem 5.6.5)

13. When the state police measure vehicle speed from aircraft, one approach of determining a car's speed is to time how long it takes to travel a fixed distance, say a quarter mile. Suppose that you were timed and the state police recorded 11.25 seconds. They charge you with speeding at 80 mph. If the police never actually recorded your exact speed, how can they guarantee that you must have been speeding? (Hint: Use Theorem 12.5.5

Part II