9.4 The Derivative of Exponential Functions

We learned that the elementary exponential functions are of the form

$$\exp_b(x) = b^x$$

for positive real numbers b. Because exponential functions involve powers, a common mistake students make is to use the power rule of derivatives. That rule only applies to power functions, where the independent variable is the base and the exponent is a constant. We will need a new rule for exponential functions.

In this section, we explore the derivative rule associate with exponential functions. As this is a new rule, we return to the definition of the derivative. We will learn that the derivative of an exponential function is proportional to the value of the function itself. The constant of proportionality is found using the base of the exponential.

9.4.1 Elementary Exponential Functions

The definition of the derivative allows us to develop a new differentiation rule. The key property necessary for the exponential function is a consequence of the properties of exponents,

$$\exp_b(x+y) = b^{x+y} = b^x \cdot b^y = \exp_b(x) \cdot \exp_b(y).$$

Using these properties

$$\begin{aligned}
\exp_b'(x) &= \lim_{h\to 0} \frac{\exp_b(x+h) - \exp_b(x)}{h} \\
&= \lim_{h\to 0} \frac{b^{x+h} - b^x}{h} \\
&= \lim_{h\to 0} \frac{b^x b^h - b^x}{h} \\
&= \lim_{h\to 0} \exp_b(x) \cdot \frac{b^h - 1}{h} \\
&= \exp_b(x) \cdot \lim_{h\to 0} \frac{b^h - 1}{h}.
\end{aligned}$$

This means that the derivative of b^x is just b^x times some number $L(b)$ that depends on b,

$$\frac{d}{dx}[b^x] = b^x \cdot L(b),$$

where $L(b)$ is calculated as the limit

$$L(b) = \lim_{h\to 0} \frac{b^h - 1}{h}.$$

Unfortunately, this is not a limit that we know as of yet. The following table illustrates some approximations for the value of this limit for a variety of bases.

h	$\frac{2^h-1}{h}$	$\frac{3^h-1}{h}$	$\frac{5^h-1}{h}$	$\frac{10^h-1}{h}$
$h = 0.1$	0.71773	1.16123	1.74619	2.58925
$h = 0.01$	0.69555	1.10467	1.62246	2.32930
$h = 0.001$	0.69339	1.09922	1.61073	2.30524
$h = 0.0001$	0.69317	1.09867	1.60957	2.30285
$h \to 0$	0.69315	1.09861	1.60944	2.30259

Every positive real number b has such a limit, $L(b)$. This limit corresponds to the slope of the elementary exponential function $y = b^x$ at the point $x = 0$,

$$L(b) = \exp_b'(0) = \lim_{h \to 0} \frac{b^{0+h} - b^0}{h} = \lim_{h \to 0} \frac{b^h - 1}{h}.$$

For the four bases used above, this corresponds to the following derivatives:

$$\begin{aligned} \frac{d}{dx}[2^x] &= L(2) \cdot 2^x \approx 0.69315 \cdot 2^x \\ \frac{d}{dx}[3^x] &= L(3) \cdot 3^x \approx 1.09861 \cdot 3^x \\ \frac{d}{dx}[5^x] &= L(5) \cdot 5^x \approx 1.60944 \cdot 5^x \\ \frac{d}{dx}[10^x] &= L(10) \cdot 10^x \approx 2.30259 \cdot 10^x \end{aligned}$$

We can see from the table that $L(2) \approx 0.69315$ and $L(3) \approx 1.09861$. This suggests that there is a particular base b between 2 and 3 such that $L(b) = 1$. Such a value does exist, using $b \approx 2.71828183$. This value has the property that the elementary exponential function and its derivative are exactly equal. The base is called the **natural base** and is given the special symbol e. Consequently,

$$\frac{d}{dx}[e^x] = e^x.$$

Definition 9.4.1 The number e is that positive value such that

$$\lim_{h \to 0} \frac{e^h - 1}{h} = 1.$$

$\Diamond$

Theorem 9.4.2

$$\frac{d}{dx}[e^x] = e^x$$

Every exponential function is proportional to its derivative. This means that at every point on the graph $y = b^x$, the ratio of the slope to the y-value is always the same constant. The interactive graph in Figure 9.4.3 illustrates this principle. The proportionality constant, $L(b)$, has been defined by a limit. We will soon discover another way to find $L(b)$ for these other bases.

Specify static image with `@preview` attribute,
Or create and provide automatic screenshot as
`images/interactive-exponential-proportional-preview.png`
via the `mbx` script

Figure 9.4.3 A graph of $y = b^x$ that illustrates the proportionality relation between the slope and the y-value.

9.4.2 The Chain Rule with Exponentials

Because the exponential function $\exp(x) = e^x$ is defined with the natural base, $\exp'(x) = e^x$ is the same as the original function, $\exp' = \exp$. Combining this with the chain rule, we get a generalized derivative of compositions with exponentials,

$$\frac{d}{dx}[e^{u(x)}] = e^{u(x)} \cdot u'(x) = e^u \cdot \frac{du}{dx}.$$

Example 9.4.4 Find the derivatives of the following functions.

1. $f(x) = e^{5x}$
2. $g(x) = e^{x^3}$
3. $h(x) = e^{x^2-4x}$

Solution. In each case, we will identify the formula $u(x)$ and then apply the chain rule.

1. For $f(x) = e^{5x}$, we have $u(x) = 5x$ so that $f(x) = e^u$. We will use $u'(x) = 5$. The chain rule gives:

$$\begin{aligned} f'(x) &= \frac{d}{dx}[e^{5x}] \\ &\underset{(u=5x)}{=} \frac{d}{dx}[e^u] \underset{(u=5x)}{=} e^u \frac{du}{dx} \\ &= e^{5x} \cdot 5 = 5e^{5x} \end{aligned}$$

So $f'(x) = 5e^{5x}$.

2. The function $g(x) = e^{x^3}$ involves a composition with $u(x) = x^3$ such that $u'(x) = 3x^2$.

$$\begin{aligned} g'(x) &= \frac{d}{dx}[e^{x^3}] \\ &\underset{(u=x^3)}{=} \frac{d}{dx}[e^u] \underset{(u=x^3)}{=} e^u \frac{du}{dx} \\ &= e^{x^3} \cdot (3x^2) = 3x^2 e^{x^3} \end{aligned}$$

Thus $g'(x) = 3x^2 e^{x^3}$.

3. The function $h(x) = e^{x^2-4x}$ involves a composition with $u(x) = x^2 - 4x$ such that $u'(x) = 2x - 4$.

$$\begin{aligned} h'(x) &= \frac{d}{dx}[e^{x^2-4x}] \\ &\underset{(u=x^2-4x)}{=} \frac{d}{dx}[e^u] \underset{(u=x^2-4x)}{=} e^u \frac{du}{dx} \\ &= e^{x^2-4x} \cdot (2x-4) = (2x-4)e^{x^2-4x} \end{aligned}$$

Thus $h'(x) = (2x-4)e^{x^2-4x}$.

□

9.4.3 Other Exponential Bases

Every function involving a positive base raised to an exponent can be rewritten using the natural exponential function $\exp(x) = e^x$. Recall that the exponential and the logarithm are inverse functions so that for every number $u > 0$ we have

$$\exp(\ln(u)) = e^{\ln(u)} = u.$$

This identity and the rules for logarithms show that any power u^w with $u > 0$ can be rewritten as

$$u^w = \exp(\ln(u^w)) = e^{\ln(u^w)} = e^{w \cdot \ln(u)}.$$

Example 9.4.5 Rewrite each of the following in terms of the natural exponential function.

1. $f(x) = 2^x$
2. $g(x) = 5^x$
3. $p(x) = x^{3/4}$
4. $r(x) = x^x$

Solution.

1. $f(x) = 2^x = \exp(\ln(2^x)) = e^{\ln(2^x)} = e^{x \ln(2)}$
2. $g(x) = 5^x = \exp(\ln(5^x)) = e^{\ln(5^x)} = e^{x \ln(5)}$
3. $p(x) = x^{3/4} = \exp(\ln(x^{3/4})) = e^{\ln(x^{3/4})} = e^{\frac{3}{4} \ln(x)}$
4. $r(x) = x^x = \exp(\ln(x^x)) = e^{\ln(x^x)} = e^{x \ln(x)}$

□

Because every exponential can be rewritten in terms of the natural exponential, we have a special result for all other exponential functions,

$$b^x = e^{x \cdot \ln(b)}.$$

Using the chain rule with $u = x \cdot \ln(b)$ and $\frac{du}{dx} = \ln(b)$, we discover

$$\frac{d}{dx}[b^x] = \frac{d}{dx}[e^{x \ln(b)}] = e^{x \ln(b)} \cdot \ln(b) = b^x \cdot \ln(b).$$

Because we already had a formula $\frac{d}{dx}[b^x] = L(b) \cdot b^x$, we now have an exact expression for the limit $L(b)$:

$$L(b) = \lim_{h \to 0} \frac{b^h - 1}{h} = \ln(b).$$

Theorem 9.4.6

$$\frac{d}{dx}[b^x] = b^x \cdot \ln(b)$$

9.4.4 Power Function or Exponential Function

One of the challenges for a calculus novice is identifying which rule applies. It is essential that you can distinguish between an exponential function and a power function.

Recall that a power function has a constant power while an exponential function has a constant as the base. Furthermore, don't be fooled by numbers that look like other symbols. For example, x^e is a power function since the power is the constant value e (the natural exponential base):

$$\frac{d}{dx}[x^e] = ex^{e-1}.$$

Similarly, $x^{\sqrt{2}}$ is a power function because the power $\sqrt{2}$ is a number (even though it is represented as a formula, it does not have any variables):

$$\frac{d}{dx}[x^{\sqrt{2}}] = \sqrt{2}x^{\sqrt{2}-1}.$$

Furthermore, you must look at a formula and determine which operation determines the differentiation rule that is required (constant multiple, sum, product, quotient or chain). This is always based on the last operation to be applied under the rules of order of operations. When the number of steps is small, you should be able to write the derivative down directly. When the number of steps is large, you might need to use the differentiation operator to allow yourself the chance to work part of the way and indicate that there are still steps remaining.

Example 9.4.7 Find the derivative $\frac{d}{dx}[(x^2+4)^5 e^{5x^3}]$.

Solution. Start by identifying the final operation. In this problem, the function $f(x) = (x^2+4)^5 e^{5x^3}$ is a product of $(x^2+4)^5$ and e^{5x^3}. So we begin by using the product rule. If you want to emphasize this without having to write down the derivatives of the factors, then we use the differentiation operator:

$$f'(x) = \frac{d}{dx}[(x^2+4)^5] \cdot e^{5x^3} + (x^2+4)^5 \cdot \frac{d}{dx}[e^{5x^3}].$$

Notice that the differentiation operator is pointing out where we still need to find derivatives in order to complete the problem.

The first term $(x^2+4)^5$ should be recognized as a composition with a power function u^5 where $u = x^2+4$. We will use the chain rule:

$$\frac{d}{dx}[(x^2+4)^5] \underset{(u=x^2+4)}{=} \frac{d}{du}[u^5] \cdot \frac{du}{dx} = 5(x^2+4)^4 \cdot (2x).$$

The second term e^{5x^3} should be recognized as a composition with the exponential function e^u where $u = 5x^3$. Again, the chain rule guides us:

$$\frac{d}{dx}[e^{5x^3}] \underset{(u=5x^3)}{=} \frac{d}{du}[e^u] \cdot \frac{du}{dx} = e^{5x^3} \cdot (15x^2).$$

The previous paragraph represents work that you either think through mentally or write out as scratch work. Putting the pieces together gives us the overall answer.

$$\begin{aligned} f'(x) = \frac{d}{dx}[(x^2+4)^5 e^{5x^3}] &= \frac{d}{dx}[(x^2+4)^5] \cdot e^{5x^3} + (x^2+4)^5 \cdot \frac{d}{dx}[e^{5x^3}] \\ &= 5(x^2+4)^4(2x) \cdot e^{5x^3} + (x^2+4)^5 \cdot e^{5x^3} \cdot (15x^2) \\ &= 10x(x^2+4)^4 e^{5x^3} + 15x^2(x^2+4)^5 e^{5x^3} \end{aligned}$$

As we start to use our derivatives in applications, we will often need to factor our formulas. To illustrate this principle, we identify all of the common factors of the terms.

$$\begin{aligned} f'(x) &= 5x(x^2+4)^4 e^{5x^3} \cdot (2+3x(x^2+4)) \\ &= 5x(x^2+4)^4 e^{5x^3}(2+12x+3x^2) \end{aligned}$$

□

9.4.5 Another Limit Defining e

When we defined the number e, it was done so that $\frac{d}{dx}\Big[e^x\Big] = e^x$. The natural base was then chosen so that

$$\lim_{x\to 0} \frac{e^x - 1}{x} = 1.$$

Suppose that we instead looked for a function $f(x)$ so that $f(0) = 1$ and $f'(x) = f(x)$. An equation for an unknown function involving derivatives is called a differential equation. We already know that $f(x) = e^x$ is the solution to this differential equation. However, we will use the differential equation to gain some additional insights into the function.

We begin by considering how we might approximate our function using only the differential equation. Consider the interval $[0, x]$ and create a partition with n subintervals,

$$x_k = \frac{kx}{n}.$$

We will recursively calculate a sequence of values $y_k \approx f(x_k)$. From the definition of an accumulation function, we know

$$f(x_{k+1}) = f(x_k) + \int_{x_k}^{x_{k+1}} f'(z)\,dz.$$

We further know $f'(z) = f(z)$ by our differential equation. So if $\Delta x = \frac{x}{n}$ is sufficiently small, we can approximate $f'(z)$ on the interval $[x_k, x_{k+1}]$ by a constant value $f'(z) \approx f'(x_k)$. This method for approximating the next value of the function defined by a differential equation,

$$f(x_{k+1}) \approx f(x_k) + f'(x_k)(x_{k+1} - x_k) = f(x_k) + f'(x_k)\Delta x,$$

is called the **Euler method**.

Our differential equation $f'(x) = f(x)$ allows us to find a value for $f'(x_k) = f(x_k)$. For our differential equation, the Euler method approximation gives us

$$f(x_{k+1}) \approx f(x_k) + f(x_k)(x_{k+1} - x_k) = f(x_k) \cdot (1 + \Delta x).$$

The approximation is then characterized by the recursive sequence,

$$\begin{aligned} y_0 &= 1, \\ y_{k+1} &= y_k(1 + \Delta x), \end{aligned}$$

which we identify as a geometric sequence with explicit formula

$$y_k = (1 + \Delta x)^k.$$

We also know that $f(x) = e^x$ is the solution. Since $x_n = x$, the Euler method approximation shows that $f(x_n) = e^x$ will be approximated by $y_n = (1 + \Delta x)^n$. Because $\Delta x = \frac{x}{n}$, we can write $n = \frac{x}{\Delta x}$ to obtain

$$f(x) = e^x \approx (1 + \Delta x)^{x/\Delta x} = \Big((1 + \Delta x)^{1/\Delta x}\Big)^x.$$

It would appear that e can be approximated by

$$e \approx (1 + \Delta x)^{1/\Delta x}$$

when Δx is sufficiently small. We state this heuristic result as the following unproved theorem.

Theorem 9.4.8

$$e = \lim_{h \to 0} (1+h)^{1/h} = \lim_{n \to \infty} (1 + \tfrac{1}{n})^n.$$

We don't have a proof at this point because without first knowing the function e^x, we do not have a clear definition for nonrational powers to justify the following limit steps:

$$\begin{aligned} f(x) &= \lim_{\Delta x \to 0} \left[\left((1+\Delta x)^{1/\Delta x} \right)^x \right] \\ &= \left(\lim_{\Delta x \to 0} \left[(1+\Delta x)^{1/\Delta x} \right] \right)^x \\ &= e^x. \end{aligned}$$

Mathematically, the best way to prove that these limits are valid is actually to work with the inverse function, the natural logarithm.