9.5 Summary

- Given any positive base $b > 0$, we know

$$\frac{d}{dx}[b^x] = b^x \cdot L(b)$$

where $L(b)$ is defined by a limit

$$L(b) = \lim_{x \to 0} \frac{b^x - 1}{x}.$$

- The number e is the natural base such that $L(e) = 1$,

$$\lim_{x \to 0} \frac{e^x - 1}{x} = 1.$$

Consequently,

$$\frac{d}{dx}\left[e^x\right] = e^x.$$

- The general derivative rule for exponentials applies the chain rule,

$$\frac{d}{dx}\left[e^u\right] = e^u \cdot \frac{du}{dx}.$$

- Expressing other formulas involving powers in terms of e,

$$u^v = e^{v \cdot \ln(u)},$$

we can show

$$\frac{d}{dx}\left[b^x\right] = \frac{d}{dx}\left[e^{x\ln(b)}\right] = e^{x\ln(b)} \cdot \ln(b).$$

This also shows that

$$\lim_{x \to 0} \frac{b^x - 1}{x} = \ln(b).$$

- The function $f(x) = e^x$ is the solution to a differential equation $f'(x) = f(x)$ with $f(0) = 1$.

- $\lim\limits_{h \to 0}(1+h)^{1/h} = e$ and $\lim\limits_{n \to \infty}(1 + \frac{1}{n})^n = e$.

9.5.1 Exercises

Foundations

1. Use the definition to find $L(\frac{1}{2})$ using a table to approximate the limit. Compare the result to $\ln(\frac{1}{2})$.

2. Use the definition to find $L(4)$ using a table to approximate the limit. Compare the result to $2\ln(2)$.

3. Find the tangent line of $y = 3^x$ at $x = 0$, using an exact expression.

Find the indicated derivatives.

4. $\frac{d}{dx}\left[e^{4x}\right]$

5. $\dfrac{d}{dx}\left[3^{2x}\right]$

6. $\dfrac{d}{dt}\left[5e^{-3t}\right]$

7. $\dfrac{d}{dx}\left[\dfrac{3}{e^{\frac{1}{2}x}}\right]$

8. $\dfrac{d}{ds}\left[\dfrac{1}{3e^{-5s}}\right]$

9. $\dfrac{d}{dx}\left[x \cdot 2^{-x}\right]$

10. $\dfrac{d}{dx}\left[e^{x^2}\right]$

11. $\dfrac{d}{dx}\left[e^{\sqrt{x}}\right]$

12. $\dfrac{d}{dx}\left[e^{e^x}\right]$

13. $\dfrac{d}{dx}\left[e^{(2x+1)^4}\right]$

14. $\dfrac{d}{dx}\left[(e^{3x} - 1)^5\right]$

15. $\dfrac{d}{dt}\left[\dfrac{2}{e^{-t} + 1}\right]$

16. $\dfrac{d^2}{dx^2}\left[4xe^{-2x}\right]$

17. $\dfrac{d^2}{dx^2}\left[x^2 e^{5x}\right]$

18. $\dfrac{d^2}{dx^2}\left[e^{-x^2+3x}\right]$

Differential Equations

19. Show that $y(t) = Ae^{kt}$, where A and k are constants, is a solution to the differential equation $\frac{dy}{dt} = ky$ for any value of A. That is, using the proposed formula for $y(t)$, compute $\frac{dy}{dt}$ and $k \cdot y$ and show that they are equal.

20. Find a solution for the differential equation $\frac{dy}{dt} = 2y$ with an initial value $y(0) = 200$. Use the proposed formula from Exercise 9.4.7.19 and solve for the value A which also satisfies the initial value.

21. A population grows at a rate that is proportional to the current population size,

$$\frac{dP}{dt} = k \cdot P.$$

If the population P is currently 2000 individuals and is growing at an instantaneous rate of 40 individuals per day, find the value k and solve the differential equation. Use the proposed formula from Exercise 9.4.7.19. What will be the population size in one week?

22. A radioactive substance decays at a rate that is proportional to the current mass of the substance,

$$\frac{dM}{dt} = -k \cdot M.$$

If the mass M is currently 50 milligrams and is decaying at an instantaneous rate of 2 micrograms per second, find the value k and solve the differential equation. Use the proposed formula from Exercise 9.4.7.19. What will be the mass of the radioactive substance after one day?